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Preface

This book was born somewhat out of necessity. AP Precalculus[1], the course, was new this year, 2023-2024. No one had many resources. All anyone had to guide them as they prepared lessons and assessments was the Course and Exam Description (CED) for the course.

In the process of teaching the course and writing my own materials this year, I found myself poring through the CED, AP Classroom, and resources shared by others to the point that I came to know the course and, as a result, the exam, quite intimately. Though an AP Exam for AP Precalculus has not even been offered yet, the level of detail about the exam that is publicly available makes writing practice materials more accessible than I thought previously imaginable. While other courses may require years of seeing released free response questions before similar high-quality questions can be written, the AP Precalculus free response questions are heavily scripted. The multiple choice section comes with clearly delineated breakdowns of content, skills, and presentation.

When I first thought about what I was going to do for AP exam review, I of course considered purchasing a set of review books from one of the major test prep companies. Then, I thought, if I know the material so well, have taught the course for as long as anyone else, and written lessons and assessments for my students all year, why not write one myself?

Thus, the book was born.

I wrote this book between November of 2023 and February of 2024, constantly thinking of how I could weave the tremendous amount of content in this course into a few weeks' worth of sweeping lessons. The goal of this book is for a teacher to be able to work with their students through the seven lessons, have their students practice with multiple choice and free response questions similar to those that will be on the AP exam, and ultimately help students "ace" the AP Precalculus exam.

I would like to thank my wife, MaryBeth, for her unwavering support, and my own AP Precalculus students this year. They are the real pioneers, and their experiences with this course are what informed so much of what has gone into this book. Without them, this book would be meaningless to me.

David Hornbeck
February 2024

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Introduction

There are a few things to know about this book and how it's intended to be used.

Structure of the Book

The lessons are based on 90-minute class periods, but they could easily be split up into 45- or 50-minute chunks as needed. In my own classroom, I plan to be at the board, working through all of the examples with students as they record the solutions in their own notes. The structure of the seven lessons are meant to spiral students through the course material in a way similar to what they can expect on the AP exam. The exam won't explicitly cover Unit 1 to Unit 3 in order, and so neither should a review book. That said, organizing into convenient "big ideas" was more difficult than I thought. The book starts with arguably the biggest idea in the course - rates of change - and from there attempts to mix in material from all different units into some of the major concepts and skills covered in the course (with an exception of Lesson 5, which explicitly covers Unit 3).

The number seven for the lessons in this book was born out of my own scheduling constraints. I see my students either twice or thrice a week for 90 minutes, so a month of review corresponds to only 10 or 11 class periods. Some of these days may be used for giving partial mock exams, and others will see students out for field trips, assemblies, or other AP exams. I ended up settling on seven lessons so as to give myself some flexibility.

Mini-AP Exams

Each lesson comes with a "mini AP-exam." Unlike the mock exams in this book, these are not explicitly designed to satisfy any content breakdowns. Some questions may even appear somewhat out of place with regards to the lesson. This was an inevitable consequence of trying to cover so much material per lesson. Rather than limit the mini-exams to the exact types of problems covered in the lessons, though, I decided to keep certain questions that would push students or seem not entirely related to the lesson. The AP exam will throw curveballs at students, so I wanted this book to do the same.

Solutions

The mini-exams lead to an important note about this book: there is a separate solutions manual. Ideally, a teacher using this book could use the mini- or mock exams as actual classroom assessments (what gets students to study more than giving a test?). Providing solutions in this book would make this impossible, so I decided to include all of the solutions to the mini-exams, mock exams, and skill practice in a separate manual.

To discourage students from buying the solutions themselves, I have made the publicly searchable solutions manual prohibitively expensive. However, if a teacher were to buy 10 or more copies of this book, there will be a private link where the solutions manual will be sold for $5, less than the cost to takes to print the book. Simply e-mail me a receipt at dhornbeck@rockdale.k12.ga.us and I will send this private link.

Mock Exams

Mock exams are a prized commodity in the AP world, but their quality can vary wildly. This book comes with two of them, but why should you trust them?

I cannot claim these exams are perfect by any stretch of the imagination, but what I can assure you of is that they have been designed to meet all of the specifications of the AP exam that have been released publicly. Each exam was crafted to satisfy the percentages of function types, Skills, and calculator skills and usage. The presentation types - graphical, analytical, tabular, and verbal - were varied similarly to what can be expected on the exam. The free response questions were crafted carefully and thoroughly to mimic as closely as possible what students should expect to see on the exam.

It was my intent that these mock exams be ever so slightly more difficult than what I anticipate the actual AP exam will be. In my experience, I have found that over-preparing students and possibly giving them a conservative impression of what they will score is far more beneficial than its optimistic counterpart. The cut scores you will find in the solutions manual are mere approximations, but I hope that they end up being a bit higher than those on the actual exam.

Appendices
In addition to the seven lessons, I decided to add in appendices that I thought would be useful for my students. In order to succeed on the AP exam, students will need proficiency with certain skills and a deep knowledge of numerous function families. Students will also need to know certain trigonometric identities, and it's important that they know exactly what identities they'll be responsible for. It is my hope that these appendices could be used by teachers as additional review lessons or supplementary homework/practice. It is worth noting that answers to Appendix D are available in the solutions manual.

Calculators
This book makes the assumption that students are working with a TI-84 graphing calculator. This was both a selfish decision, as my own students have access to a class set of these calculators, but also a practical one. If I included instructions for and solutions based on TI-Nspires, various Casio models, and the Numworks graphing calculator, the book would become substantially longer, more expensive, and even possibly give the impression that the calculator is of the utmost importance.

LESSON 1

Rates of Change

AP Precalculus is, among other things, a course about functions and the relationships between their inputs and outputs. From linear all the way through polar, functions describe how inputs and outputs change together. This idea is called *covariation*.

Throughout the course (and this book), the relationships between inputs and outputs of functions were expressed in a variety of ways: analytically, graphically, verbally, and tabularly. As you work through this book, it will be vital that you're comfortable with the varying representations of the concepts.

One of the key concepts when examining how inputs and outputs change together is **rate of change**, which describes how the outputs are changing as the inputs change. Given a function $y = f(x)$ and any two input-output pairs $(x_1, f(x_1))$ and $(x_2, f(x_2))$, the change in the outputs is $f(x_2) - f(x_1)$. The rate of change over the **interval** $x_1 \le x \le x_2$ can therefore be approximated by the change in outputs divided by the change in inputs or

$$\text{Rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

This quotient is called the **average rate of change**.

If this equation seems familiar, it is because it is the **slope** of the line going through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. The slope of this line, called a **secant line** for a function, is the graphical interpretation of the average rate of change. There are, of course, other representations, as shown in Figure 1.1.

When the rate of change of a function is positive, we say the function is **increasing**; when the rate of change is negative, we say the function is **decreasing**.

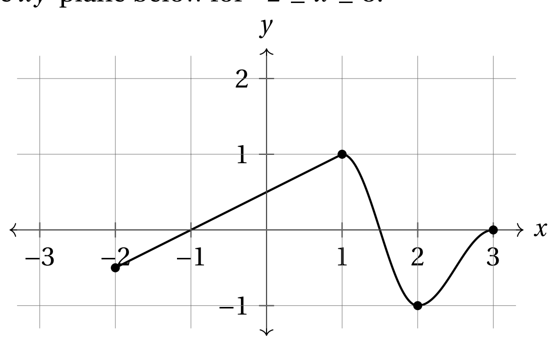
Example 1.1: Rates of Change Graphically

Let $f(x)$ be the function shown in the graphed in the xy-plane below for $-2 \le x \le 3$.

(a) On what interval/s is f increasing?

(b) Find the average rate of change of f on the interval $-2 \le x \le 3$.

(c) On which interval, $[-2, 1]$, $[1, 2]$, or $[2, 3]$, is the average rate of change of f the greatest?



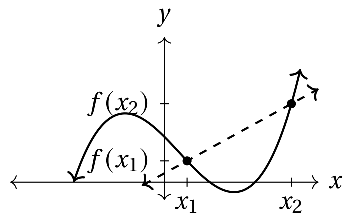
<table>
<tr><th colspan="2">Average Rate of Change (ARC)</th></tr>
<tr><td>Analytical</td><td>$\dfrac{\Delta f}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$</td></tr>
<tr><td>Graphical</td><td>The slope of the secant line

</td></tr>
<tr><td>Tabular</td><td><table><tr><th>x</th><th>$f(x)$</th><th>Difference</th></tr><tr><td>x_1</td><td>$f(x_1)$</td><td></td></tr><tr><td>x_2</td><td>$f(x_2)$</td><td>$f(x_2) - f(x_1)$</td></tr></table>
The ARC is the ratio of the difference in outputs to the length of the interval, $x_2 - x_1$.</td></tr>
<tr><td>Verbal</td><td>From x_1 to x_2, the outputs changed by an average of ARC for each increase of 1 in the inputs.</td></tr>
</table>

Figure 1.1: ARC representations

Solution

(a) f is increasing when the rate of change is positive, which occurs for $-2 \le x \le 1$ and $2 \le x \le 3$.

(b) $\frac{f(3)-f(-2)}{3-(-2)} = \frac{0-\left(-\frac{1}{2}\right)}{5} = \frac{1}{10}$

(c) The average rates of change are

$$\frac{f(1)-f(-2)}{1-(-2)} = \frac{1-\left(-\frac{1}{2}\right)}{3} = \frac{1}{2} \qquad \frac{f(2)-f(1)}{2-1} = \frac{-1-1}{1} = -2 \qquad \frac{f(3)-f(2)}{3-2} = \frac{0-(-1)}{1} = 1$$

The highest average rate of change of f is 1, which occurs on the interval $[2,3]$.

Rates of change enabled us to find **extrema**, which we identified as **local** or **global** **maxima** or **minima**.

Fact 1.A: Finding Extrema

For a continuous function f, a local/relative maximum will occur

- when the function switches from increasing (positive ROC) to decreasing (negative ROC) or
- at the included endpoint with a restricted domain

Don't forget endpoints!!

Similarly, a local/relative minimum will occur

- when the function switches from decreasing (negative ROC) to increasing (positive ROC) or
- at the included endpoint with a restricted domain

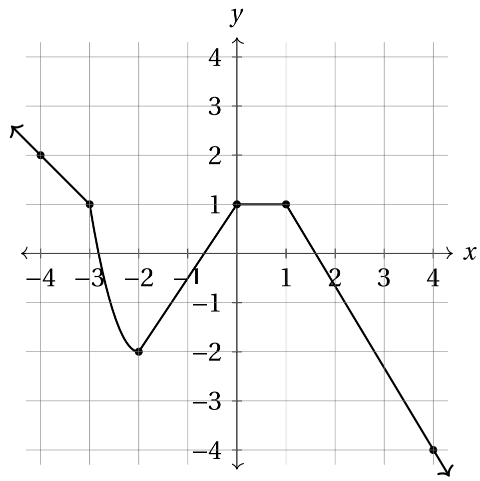
A global or absolute maximum/minimum occurs at the local maximum/minimum with the highest/lowest value among the maxima/minima, respectively.

Mini-AP Exam 1

Questions that allow a calculator will have the (▦) symbol.

Section I: Multiple Choice

1. The graph of the function f is shown. On which of the following intervals is the average rate of change of f the least?



(A) $[-4,-3]$ (B) $[-3,-2]$ (C) $[0,1]$ (D) $[1,4]$

2. The function $g(x)$ is a quadratic. Values of g are given in the table below.

x	1	4	7
$g(x)$	−5	1	10

What is $g(10)$?
(A) 20 (B) 22 (C) 28 (D) 34

3. Which of the following must be true about the polynomial function $f(x) = -3x(x-2)(x+1)$?

(A) f has a global maximum or minimum when $x = 1$.
(B) f has a global maximum or minimum when $x = c$ for some c with $0 < c < 2$.
(C) f has a local maximum or minimum when $x = 1$.
(D) f has a local maximum or minimum when $x = c$ for some c with $0 < c < 2$.

4. (▦) Let $f(x) = \frac{1}{2}x^4 - 2x^3 + \frac{3}{2}x^2 + 2x - 2$ and let $g(x) = f(x)$ for $-1 \le x \le 3$. How many total local maxima and minima does the graph of g have?
(A) Two (B) Three (C) Four (D) Five

5. The function h is given by $h(x) = -2(x-1)^2 + 3$, defined for all real numbers. Which of the following is true?

(A) The graph of h is concave down only for $x < 1$.
(B) The graph of h is concave up only for $x < 1$.
(C) The graph of h is concave down for all x.
(D) The graph of h is concave up for all x.

Section II: Free Response

x	0	1	2	3	4	5	6
$f(x)$	1	1.6	3.8	5.2	3.4	−4	−19.4

1. (🖩) The function f is defined for $x \geq 0$. The table defines the function f for select values of x within its domain. The function g is given by $g(x) = \ln(2x+4)$.

   (A) (i) The function k is defined by $k(x) = (g \circ f)(x) = g(f(x))$. Find the value of $k(0)$ as a decimal approximation, or indicate that it is not defined.

   (ii) The rate of change of f is negative for all $x > n$, where n is one of the numbers 0, 1, 2, 3, 4, 5, or 6. What is the value of n?

   (B) (i) Find all values of x, as decimal approximations, such that $g(x) = 3$.

   (ii) Determine the end behavior of g as x decreases towards $x = -2$. Express your answer using the mathematical notation of a limit.

   (C) (i) Use the table of values of $f(x)$ to determine if f is best modeled by a linear, quadratic, cubic, or exponential function.

   (ii) Give a reason for your answer based on the relationship between the change in the output values of f and the change in the input values of f.

2. (🖩) Starting in 2010, rent prices in a certain city increased rapidly. In 2010 ($t = 0$), the average rent was \$1,500, and in 2015 ($t = 5$), the average rent was \$2,700.

   The average rent in this city can be modeled by the function R given by $R(t) = a \cdot b^t$, where $R(t)$ is the average rent in year t, and t is the years since 2010.

   (A) (i) Use the given data to write two equations that can be used to find the values for constants a and b in the expression for $R(t)$.

   (ii) Find the values for a and b as decimal approximations.

   (B) (i) Use the given data to find the average rate of change of the average rent, in dollars per year, from $t = 0$ to $t = 5$ years.

   (ii) Use the average rate of change found in (i) to estimate the average rent in this city for $t = 7$ years. Show the work that leads to your answer.

   (iii) The average rate of change found in (i) can be used to estimate the average rent price in this city during year t for $t > 7$ years. Will these estimates, found using the average rate of change, be less than or greater than the average rent price predicted by the model R during year t for $t > 7$ years? Explain your reasoning.

   (C) A bill was enacted in 2021 that aimed to reduce the rapidly increasing rent prices in this city. Based on this information, what is a reasonable domain of the function R?