Periodic Rolling and Bumping

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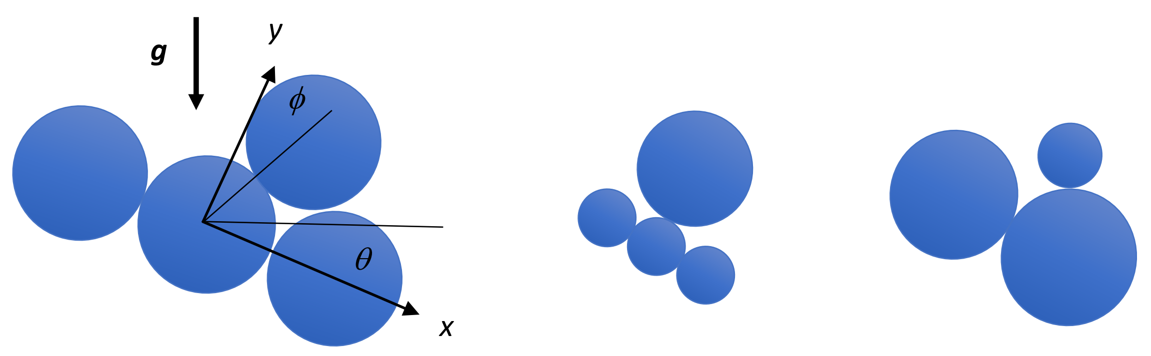
Model

A sphere of radius *R*, mass *m,* and moment of inertia *I* rolls over a corrugated surface inclined to the horizontal at an angle θ composed of contacting circular cylinders of radii *r*. The gravitational acceleration is *g*, coefficient of sliding friction is μ, components of the velocity vector $\boldsymbol{v}$ of the center of the moving sphere parallel and perpendicular and to the line of centers of the fixed cylinders are *u* and *v*. Both *u* and *v* are functions of coordinate *x* along the line of centers, measured from the center of a fixed disk; transverse coordinate *y*, measured upward from the same center.



The periodic geometry, when $R = r$, and two radii differences: $R/r = 2$ and the $R/r = 1/2$. The maximum value of ϕ, $\phi' \equiv \sin^{-1}[1/(1+R/r)]$, determines the bumpiness. As *R/r* decreases from 2 to 1/2, the bumpiness ϕ' increases from 19.5^0 to 41.8^0.

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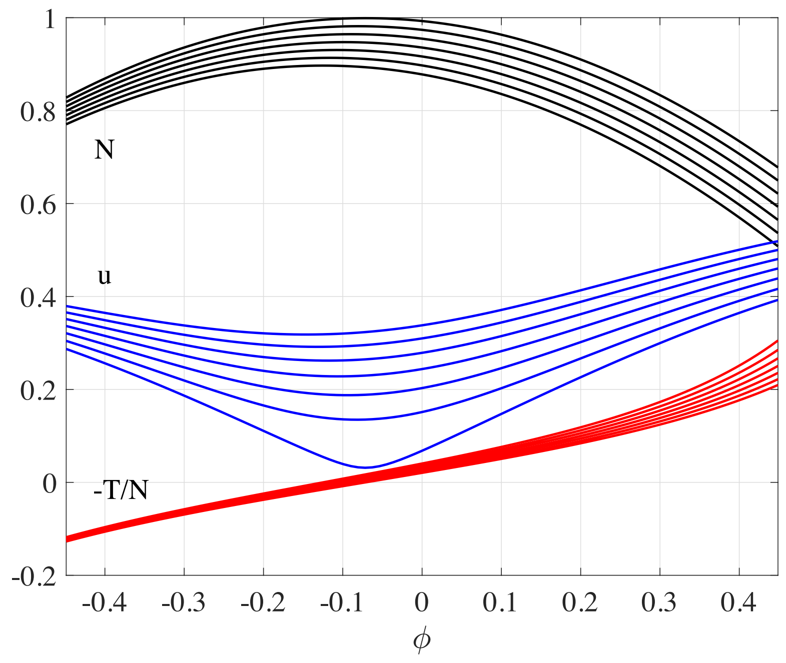
On the interval $-r \le x \le r$, along the line of centers of the fixed spheres the normal and tangential components of the force exerted by the fixed cylinder upon the moving sphere are *N* and *T*, respectively. Then, the balances of linear and angular momentum are

$$N\boldsymbol{n}+T\boldsymbol{t}+mg\langle\sin\theta,-\cos\theta\rangle = mu\frac{dv}{dx} \quad\text{and}\quad Iu\frac{d\omega}{dx} = -RT;$$

The unit vectors $\boldsymbol{n}$ and $\boldsymbol{t}$ are normal and tangent to the cylinders, and the angular velocity vector $\boldsymbol{\omega}$ is in the direction of $\boldsymbol{n}\times\boldsymbol{t}$. The dimensionless moment of inertia, $i \equiv I/(mR^2)$, is equal to 2/5 for a homogeneous sphere,

The differential equations are solved analytically when the sphere rolls without sliding for the entire periodic trajectory and when sliding occurs for some part of the trajectory. The constants of integration in the solutions are evaluated by consideration of the transfer of momentum in the bump. The sphere is assumed to maintain contact with the cylinders during the bump. The sphere may roll without sliding both before and after the bump, or roll with sliding before the bump and without sliding after.

$|T| \le \mu N$, *Rolling without sliding*



Normal force, parallel velocity, and force ratio for a homogeneous sphere at inclinations $\theta = 4.1^0$ to 7.7^0, when $\phi' = 25.71^0$ $(R/r = 1.30)$ and $\mu = 0.30$.

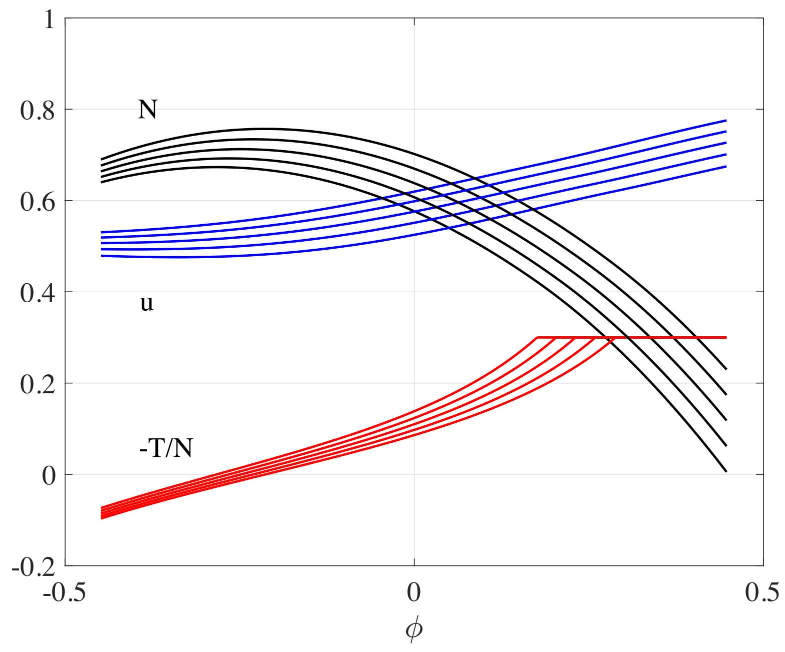
Average squared velocity

$$2\phi'\overline{u^2} = \frac{2}{1+i}\left[\left(-2\sin\phi'+\frac{2}{3}\sin^3\phi'\right)\cos\theta+\left(\frac{1+\alpha_1^2}{1-\alpha_1^2}\sin\phi'\sin\theta+\cos\phi'\cos\theta\right)\left(\phi'+\frac{1}{2}\sin2\phi'\right)\right].$$

$|T| = \mu N$, *Rolling with sliding*

The angle $\hat\phi$ at which sliding begins is determined approximately as $\hat\phi \approx \phi' - \Delta$, where

$$\left[(3+i)\sin(\phi'+\theta)+\frac{i}{\mu}\cos(\phi'+\theta)\right]\Delta = \frac{4(1+i)^2\sin\phi'\sin\theta}{(1+i)^2-\left(1+i-2\sin^2\phi'\right)^2}-(1+i)\cos(\phi'+\theta)+\frac{i}{\mu}\sin(\phi'+\theta).$$



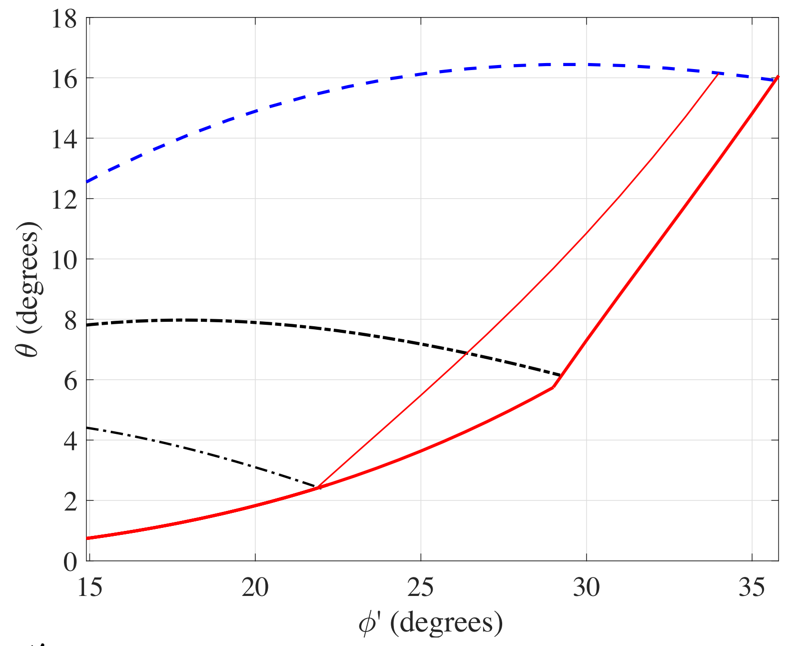
Normal force, parallel velocity, and the force ratio for a homogeneous sphere at inclinations $\theta = 12.22^0$ to 16.22^0, when $\phi' = 25.71^0$ and $\mu = 0.30$.

Average squared velocity

$$2\phi'\overline{u^2} = \frac{1}{2(1+i)}\left[(3+i)\cos(\hat\phi+\theta)-\frac{i}{\mu}\sin(\hat\phi+\theta)\right]\left(\hat\phi+\phi'+\frac{1}{2}\sin2\hat\phi+\frac{1}{2}\sin2\phi'\right)$$
$$-\frac{1}{6(1+i)}\left[6(\sin\hat\phi\cos\theta+\sin\phi'\cos\theta)+3[\sin(\hat\phi+\theta)+\sin(\phi'-\theta)]+\sin(3\hat\phi+\theta)+\sin(3\phi'-\theta)\right]$$
$$-\frac{\mu}{2(1+4\mu^2)}\left[6(\sin\phi'\sin\theta-\sin\hat\phi\sin\theta)-3[\cos(\phi'+\theta)-\cos(\hat\phi+\theta)]-\cos(3\phi'+\theta)+\cos(3\hat\phi+\theta)\right]$$
$$-\frac{(1-2\mu^2)}{6(1+4\mu^2)}\left[6(\sin\phi'\cos\theta-\sin\hat\phi\cos\theta)+3[\sin(\phi'+\theta)-\sin(\hat\phi+\theta)]+\sin(3\phi'+\theta)-\sin(3\hat\phi+\theta)\right]$$
$$+\frac{1}{4\mu(1+\mu^2)}\left[\left(\frac{6\mu}{1+4\mu^2}-\frac{i/\mu}{1+i}\right)\sin(\hat\phi+\theta)+\frac{3}{1+4\mu^2}\cos(\hat\phi+\theta)\right]$$
$$\times\left[e^{2\mu(\phi'-\hat\phi)}(\mu^2\cos2\phi'+\mu\sin2\phi'+1+\mu^2)-(\mu^2\cos2\hat\phi+\mu\sin2\hat\phi+1+\mu^2)\right].$$

Regime Transitions

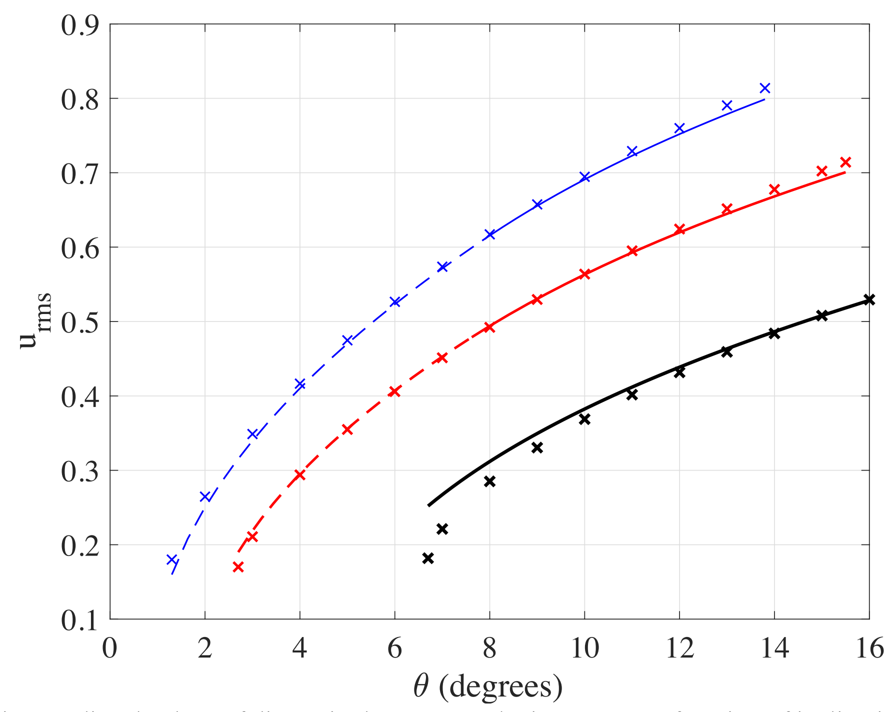
With the analytical solutions, the transitions between stoping, rolling with sliding, and loss of contact can be determined.



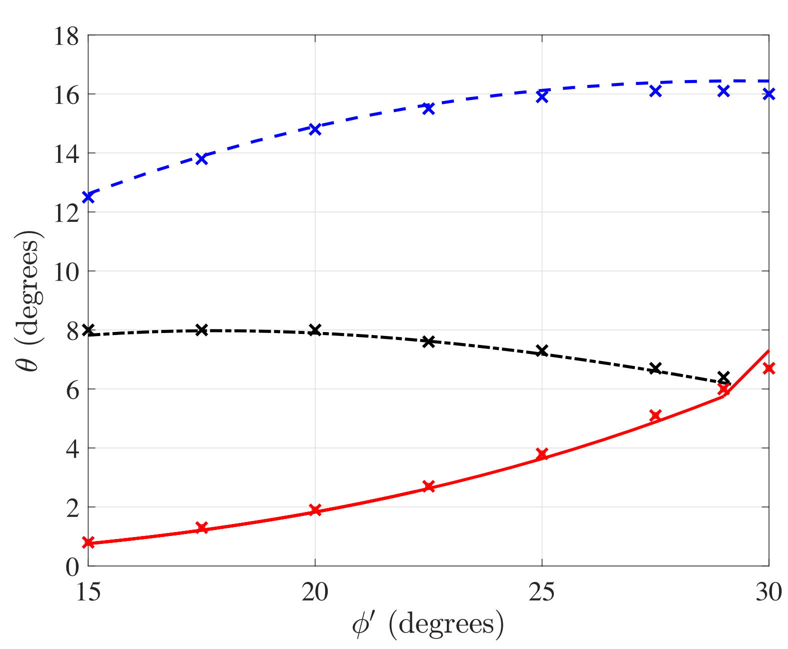
Curves of θ versus ϕ' for stopping of rolling without sliding and stopping of rolling with some sliding (solid), transitions between the two (dashed-dot), and loss of contact for rolling with some sliding (dashed) are shown for homogeneous spheres, with $\mu = 0.15$ (light) $\mu = 0.30$ (dark). For a given coefficient of friction, rolling stops below the solid red line; rolling without sliding takes place between the solid line and the dot-dashed line, there is a region of transition between the dot-dashed and dotted lines, rolling with some sliding takes place between the dotted line and the dashed line; and contact is lost above the dashed line.

Numerical Tests

The theory is tested against Discrete Element Method (DEM) simulations of a sphere of radius *R* rolling over a corrugated surface inclined to the horizontal at an angle θ composed of contacting circular cylinders of radii *r* with their axes horizontal and perpendicular to the motion of the sphere. The DEM simulation solves the equations of motion for the sphere subject to both gravity and contact forces with the cylinders of the corrugated surface.



Curves showing predicted values of dimensionless RMS velocity, u_{rms}, as a function of inclination, θ, for three values of bumpiness, ϕ' : 17.5^0 , 22.5^0, and 30.0^0, from upper to lower, and $\mu = 0.30$. Dashed regions indicate rolling without sliding; solid regions indicate rolling with some sliding. The symbols are values measured in the numerical simulations.



Transition curves of for a homogeneous sphere with $\mu = 0.30$ and values of the transitions measured in the numerical simulations (symbols).

Conclusions

The model for periodic rolling with bumping leads to differential equations that can be integrated to provide expression for the average velocity of a sphere that rolls down a corrugated surface. Constant values of this velocity are possible for ranges of inclination and bumpiness because of the energy lost in bumping. The transitions between the various modes of motion may be calculated and agree well with those seen in numerical simulations. The model is limited by the assumption that contact is not lost during a bump. The numerical simulations indicate that for coefficients of restitution as large as 0.1, the results of the calculation are unaffected; for values as large as 0.5, the motion is still periodic, but jumps are present that influence the results.