Module 6 Working with Expressions

Section 6.1 Arithmetic Operations

Looking Back 6.1

There are many ways to solve problems and get a correct answer. Sometimes, however, solutions are incorrect. These incorrect solutions are often the result of calculation errors from not following proper procedures. In mathematics, orderly steps (algorithms) are used so that everyone can find the same solution to a problem.

To compare data, we must use the same measure(s) of data for the sets at hand. In 1998, NASA spent 125-million dollars to send the Mars Orbiter around Mars with the goal of gathering information about the planet. However, the spacecraft was programmed in metric units when it was built but the engineering team used English units to guide its travel. This ended in the unfortunate destruction of the Mars Orbiter as it neared Mars' surface. The error occurred because different measurement systems were used to build and guide the Mars Orbiter. Conversions are very important!

Looking Ahead 6.1

The following three examples give three different answers for the same problem; only one answer uses the correct procedure and gives the correct answer.

Example 1: Solve (evaluate) the expression below from left to right.

$$3 \times 4 - 2 + 8 \times 6 \div 3$$

Example 2: Solve (evaluate) the expression below (the same expression from Example 1), but this time, go from left to right doing addition or subtraction first in the order they come, followed by multiplication or division in the order they come.

$$3 \times 4 - 2 + 8 \times 6 \div 3$$

Example 3: Solve (evaluate) the expression below (the same expression from Example 1 and Example 2), but this time, solve the problem from left to right doing all grouping operations first. Then solve the problem from left to right. Remember, multiplication is repeated addition and division is repeated subtraction.

$$3 \times 4 - 2 + 8 \times 6 \div 3$$

When working with the four arithmetic operations of addition, subtraction, multiplication, and division, use the following order:

Do all grouping operations first from left side to the right side. Simplify everything to addition and subtraction and continue to work from left to right.

Section 6.2 Parenthesis First

Looking Back 6.2

When there are only the four arithmetic operations: multiplication, division, addition, and subtraction, multiplication being repeated addition and division being repeated subtraction make addition and subtraction of equal importance. However, when there is parenthesis in the problem, the operations within the parenthesis take precedence over all other operations. In other words, operations inside of parenthesis come first! It is as if they shout: "Do this first!"

Remember that a variable or number next to parenthesis means to multiply. It can be in front of the parenthesis or behind them. Also remember that multiplying by a negative value changes the sign of everything within the parenthesis to the opposite sign. So, if there is a negative sign in front of the parenthesis, everything inside the parenthesis would be multiplied by -1 (because of the distributive property).

Looking Ahead 6.2

Example 1:	Evaluate the arithmetic expression below.

$$(5 + 8) - 6 \times 8 + 13$$

Example 2:	Evaluate the arithmetic expression below.

$$(2 \times 4 - 6) + (10 - 4 \div 2)$$

Example 3:	Evaluate the arithmetic expression below.

$$2(4 - 3 \times 2) - 5(4 \times 5 + 3 \times 3)$$

In summary, when performing order of operations, always begin with what is inside parenthesis. Once inside the parenthesis, simplify groupings first left to right. Continue to operate from left to right for all addition and subtraction. When finished inside the parenthesis, simplify all grouping symbols outside the parenthesis from left to right. Finally, do all remaining addition and subtraction from left to right.

Section 6.3 Exponents Next

Looking Back 6.3

Exponents are important numbers. They must be simplified to a number to be multiplied, divided, added, or subtracted to other numbers. Remember that the base is the constant or variable that is being multiplied, and the exponent tells how many times it is being multiplied. Therefore, $(-4)^2$ is -4×-4, which is equal to 16. If we put -4^2 in a calculator without parenthesis, we will get -16, which is incorrect. So, be careful squaring numbers; remember that squaring a number always results in a positive number

While parenthesis seem to shout: "Do this first!" exponents seem to shout: "Do this next!"

Looking Ahead 6.3

Example 1:	Simplify the expression below.

$$-14 \times 2^3 + 21 \div 3$$

Example 2:	Simplify the expression below.

$$-40 \times (2^4 \div 2)$$

Example 3:	Simplify the numerical expression below.

$$-10 \times 3^2 + (18 - 3) \div 3$$

Example 4:	Simplify the numerical expression below.

$$-1.5 \times 3 \times 6.2 \times 3.1^2 \times (0.7 + 0.3)$$

Example 5:	Simplify the numerical expression below.

$$-2^5 + -3^4 + (-4)^3 + (-2)^2$$

Look for parenthesis and exponents first. When those are simplified to one number, combine them with the numbers that use grouping symbols. When everything is simplified to a number, do the operations in the order they come from left to right.

Section 6.4 Order of Operations

Looking Back 6.4

Now we know that parenthesis take precedence over the other operations and simplifying exponents comes next. After doing operations inside parenthesis and then with exponents, we take care of all grouping symbols followed by operations from left to right. This will assure we all get the same answer when solving the same problems.

Mathematics is a universal language. We maintain order in problem-solving by following these specified procedures. These specified procedures are called *order of operations*. Provided there are no calculation errors, anyone in the world using the correct procedure for order of operations will get the same answer for these practice problems.

Notice that $8 + 4 \div 2$ is equal to $8 + 2 = 10$. The calculator changes "$8 + 4 \div 2 = 8 + 2 = 10$" to "$8 + \frac{4}{2} = 10$." It changes the problem to one with a grouping symbol for division.

A different problem is $\frac{8+4}{2} = \frac{12}{2} = 6$. If you put it in the calculator as "$8 + 4 \div 2$," you will get "10," which is incorrect. To correct this, put "$8 + 4$" in parenthesis so the entire numerator gets added first before it is divided by the denominator of "2." Put "$(8 + 4) \div 2$" in your calculator and it will give you the correct answer of 6.

This is why parenthesis are so important. The eye must catch them and group the things within them first. Parenthesis, and exponents as well, are grouping symbols!

Looking Ahead 6.4

Example 1:	Simplify the expression below using order of operations.

$$16 - 4^2 + (3^3 + 3) \div 10$$

Example 2:	Simplify the expression below using order of operations.

$$3^3 + 7 \times 4 + (\frac{10}{2} \times 5) \div 5 \times 1$$

Section 6.5 Algebraic Expressions

Looking Back 6.5

We have been using constants to evaluate problems with order of operations. We have also been using operations to combine like terms in algebraic expressions. These problems have been fairly simple, but now we will begin working more in depth with difficult problems.

Using order of operations to combine like terms in numerical expressions simplifies the expression to one number. We are now going to introduce problems that include variables in our algebraic expressions. When variables are involved, not all algebraic expressions have like terms, so the answer may be a monomial (one term), a binomial (two terms), a trinomial (three terms), etc. It does not become one number, but rather one term (given the variable does not cancel out).

Looking Ahead 6.5

Remember, a variable is a letter or a number with a letter used to represent a number. It is called a variable because it varies or changes. It stays the same in one expression or equation but is different in another expression or equation. Given $2x$ is equal to 10, then x is equal to 5. However, given $2x$ is equal to 12, then x is equal to 6. The x has a different value in the different equations, but it will always be 5 in the first equation ($2x = 10$), and it will always be 6 in the second equation ($2x = 12$). In an expression like $2x + 1$, $2x + 1$ is equal to 3 if x is 1, but if $2x + 1$ is equal to 7, then x is 3. The variable (x) changes.

In this section, we will learn how to write algebraic expressions before we learn to evaluate them.

Example 1: Write an algebraic expression for "8 more than a number." *More than* means to add. Assign a variable to be "a number."

Order is not important in this case because addition is commutative.

Example 2: Write an algebraic expression for "6 less than a number." *Less than* means to subtract. Assign a variable to be "a number."

Order is important in this case because subtraction is not commutative.

Example 3: Write an algebraic expression for "4.3 times a number." *Times* means to multiply. Assign a variable to be "a number."

Order is not important in this case because multiplication is commutative.

Example 4: Write an algebraic expression for "-10 divided by a number." *Divided by* means to divide. Assign a variable to be "a number."

Order is important in this case because division in not commutative.

Section 6.6 Multiple Operations

Looking Back 6.6

In the previous section, variables were used to represent numbers that can change (letters represent these), and constants were used to represent numbers that stay the same (numbers represent these). Algebraic expressions were written for phrases that involved the four mathematical operations. These were investigated one at a time. In this section, rather than investigating simple phrases, we will investigate complex phrases that represent multiple mathematical operations and require the use of order of operations.

Looking Ahead 6.6

Example 1: Write an algebraic expression for the phrase below.

"ten more than three times a number"

Example 2: Identify the variable and define it, then write an algebraic expression for each phrase below.

a) 6 less than 1.7 times a number

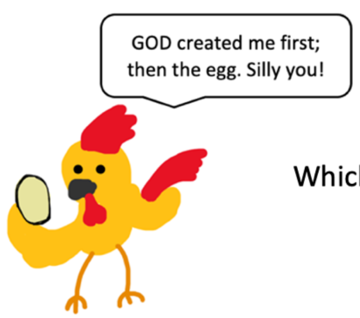
b) 4.6 more than the quotient of 11 and a number

c) the sum of a number and 12 less than that number

d) 4 less than the sum of 5 times a number plus 8

Example 3: Write an algebraic expression for the phrase below.

"12.2 more than the quotient of a number divided by 5"



Which came first, the chicken or the egg?

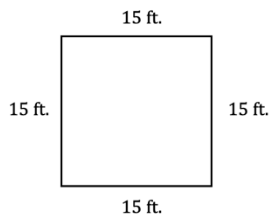
Section 6.7 Geometric Representations

Looking Back 6.7

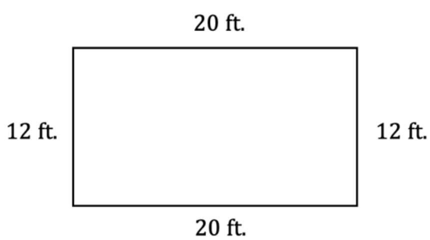
It is important to understand that mathematical concepts can be represented in many ways: numerically (which is foundational), algebraically (which uses numbers and symbols), geometrically (which uses figures), in tabular form (which uses tables), and in graphical form (which uses graphs). All of these representations can be used to analyze and solve problems. We have previously explored expressions numerically and algebraically. In this section, geometric representations of algebraic expressions will be investigated.

Looking Ahead 6.7

Example 1: Find the perimeter of a square fence that has a side of 15 feet.

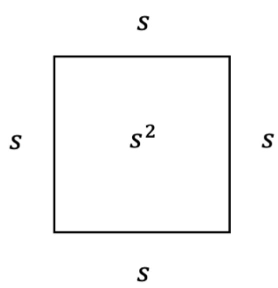


Example 2: A rectangular fence has a length of 20 feet and a width of 12 feet. Find the perimeter in more than one way.



Example 3: Draw the geometric diagram and write the algebraic expression that represents the perimeter of a rectangle that is 6 units longer than it is wide.

Example 4: Find the area of the square that is represented geometrically. What is the side length?



Example 5: Draw the geometric diagram and write the algebraic expression that represents the perimeter of a rectangle whose width is 4 units less than its length.

Section 6.8 Tabular Representation

Looking Back 6.8

In the previous section, we talked about five representations of algebraic expressions: numeric, algebraic, geometric, tabular, and graphical. Then we looked into geometric representations of algebraic expressions. In this section, we will look into tables that can be made to see patterns. In the next section, we will explore graphical representations by drawing and investigating graphs.

Looking Ahead 6.8

Example 1: The total cost of ears of corn can be found if the price for one ear is known. Complete the table below.

Number of Ears of Corn	Total Cost of Corn
1	$0.33
2	
3	
4	
n	

Example 2: Complete the table below and write an algebraic expression for the total amount of money earned when n shoes are sold.

Number of Shoes Sold	Total Amount of Money Earned
1	$21
2	$42
3	
4	
5	
n	

Example 3: In the table below, explain what the variable represents and write an expression to model it in terms of total length in feet (of boards).

Number of Boards	Total Length in Feet
1	3.2
2	6.4
3	9.6
4	12.8
$\vdots$	$\vdots$
b	

Section 6.9 Graphical Representations

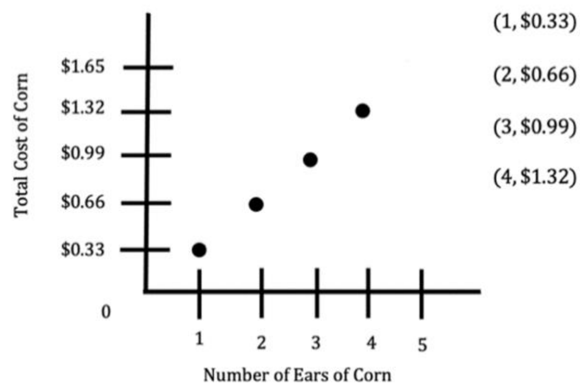
Looking Back 6.9

We have now seen the numerical, algebraic, geometric, and tabular representations of expressions. In this section, we will focus on the graphs that correspond to the data in tables.

Let us look back at a table from the previous section:

Number of Ears of Corn	Total Cost of Corn
1	\$0.33
2	\$0.66
3	\$0.99
4	\$1.32
n	\$0.33n

We can convert the ordered pairs and graph the points.

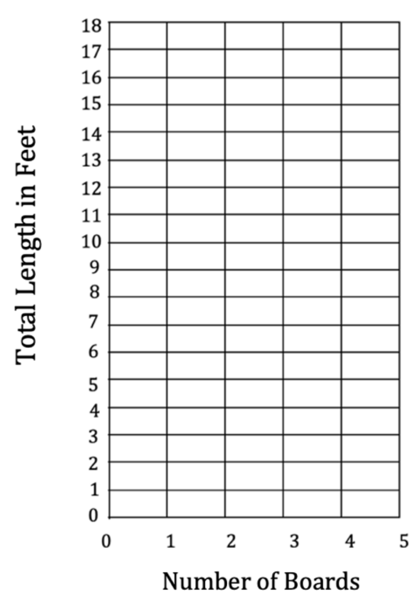


Is this graph continuous or discrete?

Looking Ahead 6.9

Example 1: Draw the graph for the table below.

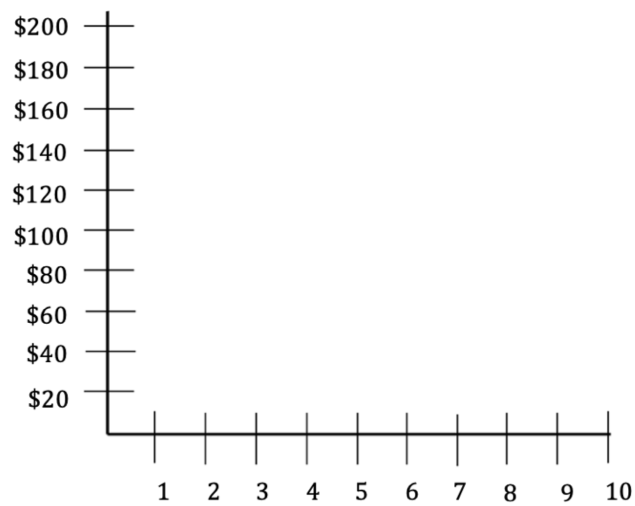
Number of Boards	Total Length in Feet
1	3.2
2	6.4
3	9.6
4	12.8
5	16.0



Example 2: We call the 3.2 feet from Example 1 the slope of the graph. What does it represent? Which is the independent variable? Which is the dependent variable? Given T is the total length in feet and n is the number of boards, write an equation to represent the relationship between the two.

Example 3: A babysitter saved money all summer to buy a 10-speed bicycle. At the end of the summer, the babysitter had $200.00 in her account, which is exactly the amount of money she needed for the bicycle. The babysitter made weekly payments of $20.00 until the bicycle was all paid for. In the table below is the week of the payment and the amount of money in the babysitter's account. Make a graph to represent the values in the table with x as the week of the payment and y as the amount of money in the account.

Week of Payment	Amount Left in Account
1	$180.00
2	$160.00
3	$140.00
4	$120.00
5	$100.00
6	$80.00
7	$60.00
8	$40.00
9	$20.00
10	$0.00



Example 4: What is the slope of the graph in Example 3? What does it represent? Which is the independent variable? Which is the dependent variable?

Section 6.10 Multiple Representations

Looking Back 6.10

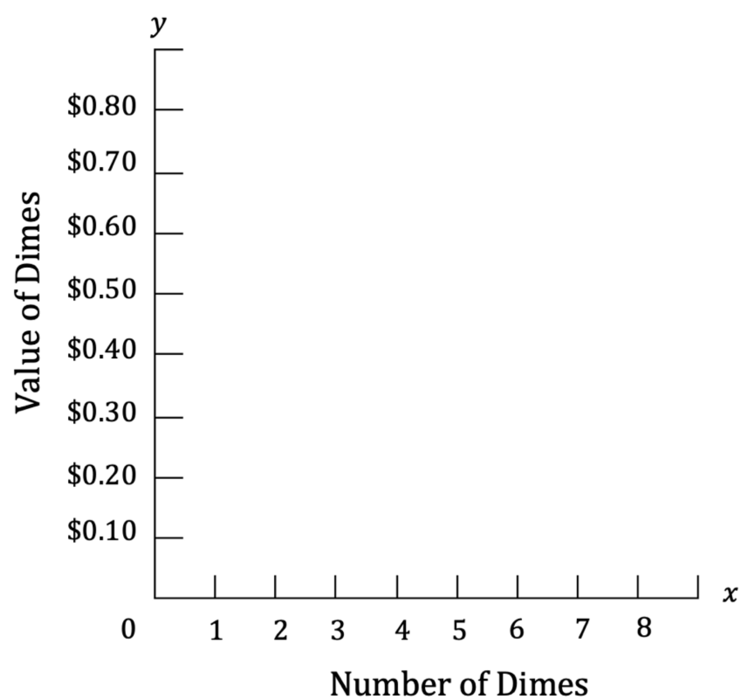
When we put our knowledge from the previous sections together, we can see that every word, phrase, or situation can be modeled by a mathematical expression or equation. There is also a numerical, algebraic, geometric, tabular, and graphical representation that models the word, phrase, or situation given. These representations will be used throughout all the high school mathematics courses. Therefore, it is important to learn to translate from one to another. Equations of higher order will be introduced as we progress through our studies and the graphs and tables that go with them will be analyzed.

God has graciously given us multiple ways to interpret and communicate the natural occurrences of creation and the world around us. These are just a few of the representations.

Looking Ahead 6.10

Example 1: Given you have a jar full of dimes, write a mathematical equation to represent how much money you have. Draw a table and graph to represent this situation. Then find the equation that represents the relationship between the number of dimes in your jar and the amount of money you have. Use your table, graph, and equation to tell how much money you have if there are 8 dimes in your jar. Let $v =$ value of dimes and $d =$ number of dimes.

Number of Dimes (d)	**Value of Dimes (v)**
1	
2	
3	
4	
5	
6	
7	
8	
d	



Section 6.11 Algebraic Equations

Looking Back 6.11

This module has been about algebraic expressions and order of operations. Algebraic expressions describe data. These expressions can be made into equations once we know what the variables represent. In a table, the independent variable (input) is the first column. In a graph, the independent variable is on the x-axis. In a table, the dependent variable (output) is the second column. In a graph, the independent variable is on the y-axis.

Looking Ahead 6.11

Example 1: Write an equation to model the total cost of canned goods given each individual canned good costs $\$0.55$. Let n be the number of canned goods and let c be the total cost.

Example 2: Write an equation to model the total number of tickets sold if 5 tickets are sold per hour. Let h be the number of hours in which tickets are sold and t be the total number of tickets sold.

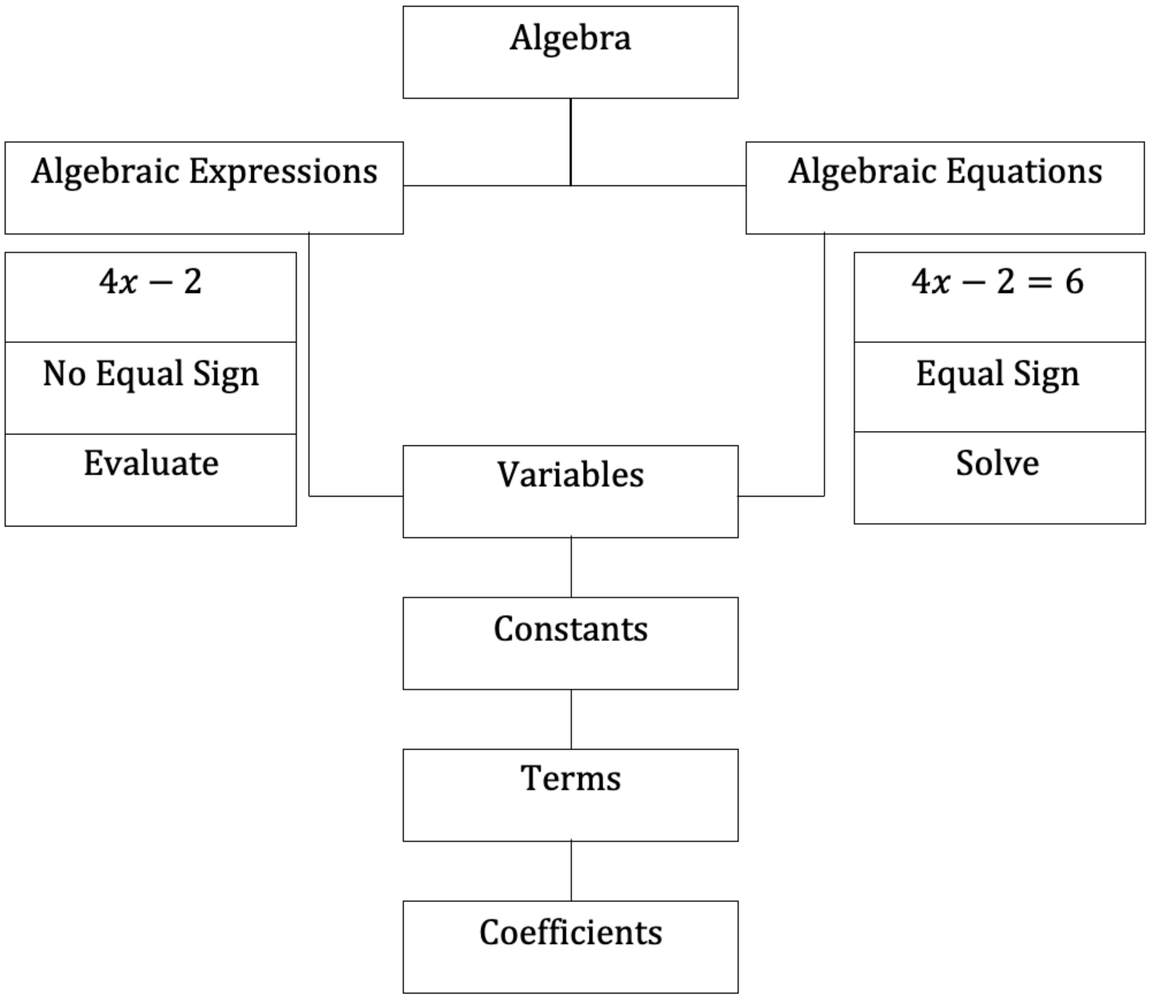
Example 3: Write an equation to represent a relationship between your total earnings and your total savings given the data. Start by making a table. Let e represent the money you earn and s represent the money you save.

Earn: $22.00 and Save: $11.00

Earn: $16.00 and Save: $8.00

Earn: $11.00 and Save: $5.50

In summary, algebraic expressions and algebraic equations have some similarities and some differences, which can be seen on the flow chart below.



Section 6.12 Evaluating Expressions

Looking Back 6.12

We have been simplifying algebraic expressions. When the terms were all numbers, the order of operations was used to get one numerical answer. In a sense, “simplify” meant to solve only in the special case that constants were present in the expression.

With algebraic expressions, like terms can be combined to get a monomial, binomial, trinomial, etc. for an answer. They can be solved if the values of the variables are known. However, because there is no equal sign in an expression to indicate a solution, the instructions often say: “Evaluate the expression,” rather than: “Solve the expression.” Again, in a sense, evaluating is solving because the result is one numerical solution. Still, we will use the term “evaluate” for our purposes.

Therefore, we are going to evaluate (solve) expressions by substituting numerical values in for variables and performing operations.

Looking Ahead 6.12

Example 1: Evaluate each algebraic expression given $a = -2$, $b = -3$, and $c = 5$.

a) $a + b^2c$

b) $(a - b)(a + c)$

c) $(ab)^2 - c$

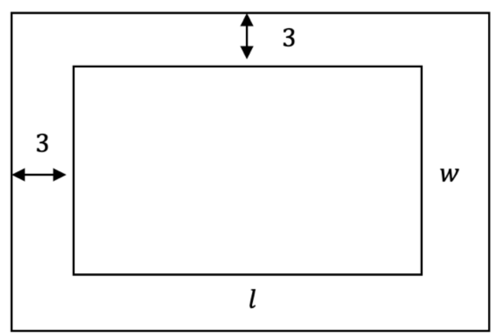
d) $\frac{b}{a} + \frac{c}{b}$

Example 2: A picture frame is 3 inches wide. If l is the length of the picture and w is the width of the picture, what is the area of the picture and frame in terms of l and w?

The length of the picture and frame is __________________.

The width of the picture and frame is __________________.

Area =



Example 3: If the length of the picture from Example 2 is 13 inches and its width is 10 inches, what is the area of the picture and frame?

Section 6.13 Problem-Solving Strategies

Looking Back 6.13

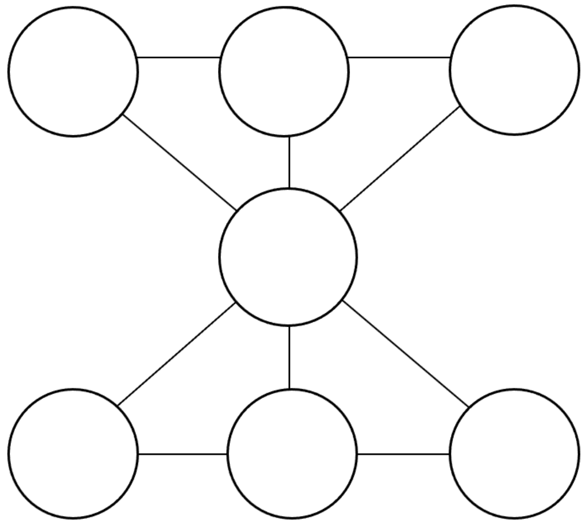
Throughout this module, we have investigated order of operations, exponents, algebraic expressions, and multiple representations for modeling mathematical problems. We have also used many problem-solving strategies to work through the mathematical challenges that have appeared in many practice problems. Some of the strategies we have used include but are not limited to: reverse thinking (working backwards), substituting and solving (guessing and checking), making a model, drawing a diagram, drawing a picture, creating a table and/or graph, and looking for patterns. Now, let us look at some of these strategies one at a time.

Looking Ahead 6.13

Example 1: A friend tells three friends a rumor on the first day. Each of those three friends tells another three friends on the second day who then each tell another three friends on the third day. How many days will it be before all 85 friends have heard the rumor? Complete the table and look for patterns in it.

Day (d)	**New Friends Who Hear the Rumor (F)**	**Total Friends Who Hear the Rumor**	**Patterns**
1			
2			
3			
4			
5			
6			
d			

Example 2: Try to put the numbers 1 through 7 in the circles below so the top and bottom rows, the middle column, and the two diagonals add up to 12. Use the Guess and Check Strategy to fill in the circles.



Example 3: A tree doubles in height each year. After 18 years, it reaches its maximum height. How many years did it take to reach half its maximum height? Use reverse thinking to figure it out.