Module 5 Working with Exponents

Section 5.1 Prime Numbers

Looking Back 5.1

We have worked with a number of problems that involve exponents. Exponents are used in prime factorization. A prime number only has two factors: 1 and itself. If a number is not prime, it is composite; a composite number has more than two factors.

Prime numbers are very interesting. They are especially useful in solving problems with fractions as we will explore.

Eratosthenes is a name we see often in mathematics. He was a mathematician in Cyrene, Greece in 276 B.C. Eratosthenes studied in Athens, which was at that time considered the center of learning throughout the world. He was accomplished in mathematics as well as astrology, using both to become the first man to determine the circumference of the earth. Eratosthenes worked extensively with prime numbers. He taught us that we can test whether a number is prime or not by dividing it by all the prime numbers smaller than itself.

In the 3rd century B.C., Euclid (another famous mathematician of antiquity) proved there are infinitely many prime numbers, though fewer as the numbers get larger. Ptolemy III (the pharaoh of Egypt at the time) invited Euclid to Egypt to tutor his son and become head librarian for the great university at Alexandria. However great his achievements, it has been passed down that Euclid's friends nicknamed him "Beta" (β), which is the second letter of the Greek alphabet because, in their opinion, he never reached his full potential.

God is often referred to by Greek monikers (nicknames). Revelation 22:13 calls God the "Alpha" (A) and the "Omega" (Ω), which are the first and last letters of the Greek alphabet. These titles mean God is the beginning and the end.

Looking Ahead 5.1

Example 1:	Tell whether 23 is prime or composite and explain why.

To test for a prime number, we can divide by all its smaller primes. In this case, they are 2, 3, 7, 11, 13, 17, 19.
None of these divides into 23 without leaving a remainder; therefore, it is prime.
Every even number greater than 2 is a sum of 2s; therefore, 2 divides into every even number without leaving a remainder, which makes all even numbers composite.

Example 2:	Tell whether 46 is prime or composite and explain why.

The number 1 is neither prime nor composite because it has only one factor, which is itself.

Example 3: Tell whether each of the numbers below are prime or composite and explain why.

a) 25

b) 110

c) 33

d) 91

In June of 1742, Christian Goldbach wrote a letter to fellow mathematician Leonhard Euler explaining his hypothesis that every number greater than 2 is a sum of two prime numbers. Although there is no proof of Goldbach's Conjecture in infinitum, it has been proven up to the zillions. No one has found even one counterexample.

Example 4: Use 3s, 5s, and 7s to prove Goldbach's Conjecture.

Goldbach's Conjecture:

$$4 = 2 + 2$$
$$6 = 3 + 3$$
$$8 = 5 + 3$$
$$10 = 7 + 3$$
$$12 = 5 + 7$$
$$14 = 7 + 7$$
$$\dots$$
$$1{,}000 = 997 + 3$$

Section 5.2 Prime Factors

Looking Back 5.2

Factors are numbers we multiply together. We have worked with them in Module 1. The Fundamental Theorem of Arithmetic states that every natural number that is a composite number has only one unique prime factorization. Therefore, the prime factorization of 14 is $2 \cdot 7$. If 1 were prime, 14 would not be unique as it could be factored primarily in several ways:

$$1 \cdot 2 \cdot 7$$

$$1 \cdot 1 \cdot 2 \cdot 7$$

$$1 \cdot 1 \cdot 1 \cdot 2 \cdot 7$$

etc.

Prime numbers can be thought of as the building blocks of all numbers. In science, atoms are the building blocks of molecules, which are the building blocks of cells. In math, prime numbers are the building blocks of other numbers.

Looking Ahead 5.2

Example 1: Find the prime factorization of 128. Write the factors as an exponent.

Example 2: Find the prime factorization of 56. Write the factors as an exponent.

Example 3: This famous cryptarithm (arithmetic problem in which letters have been substituted for numbers or vice versa) uses each of these digits: 0, 1, 2, 5, 6, 7, 8, 9. The numbers 0, 1, and 6 are used twice and 5 is used three times. The rules are as below:

a) Each letter represents one digit throughout the entire problem.

b) The math operation must stay correct when numbers replace letters.

c) Base 10 is used.

d) No word begins with "0."

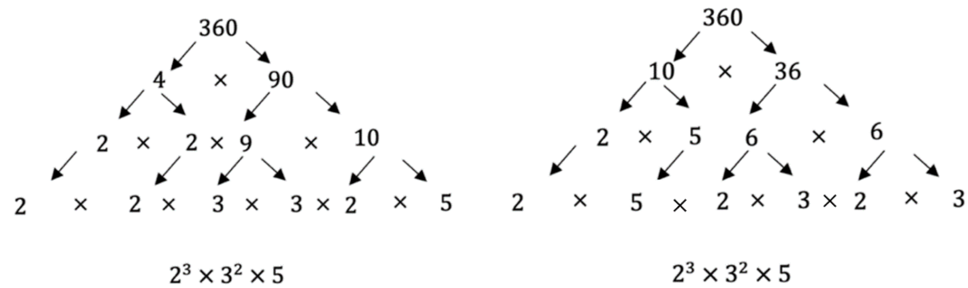
e) There is only one unique solution.

$$\begin{array}{r} \text{SEND} \\ \underline{+\ \text{MORE}} \\ \text{MONEY} \end{array}$$

Section 5.3 Prime Factors and Probability

Looking Back 5.3

The prime factorization for 360 is $2^3 \cdot 2^3 \cdot 5$. If we call the exponents a, b, and c, we have $2^a \cdot 3^b \cdot 5^c$ in which a, b, and c are positive integers. There is more than one way to find the prime factorization of 360, but the result is always the same.



All of these are factors of 360 and there are still more: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360. Any two numbers that combine for a product of 360 could be used to begin.

This is like using the "Sieve of Eratosthenes" in reverse. We know 2, 3, and 5 are factors so 2×3, 2×5, and 3×5 will also be factors. Because we know probability, we can find all the possible combinations of factors.

Looking Ahead 5.2

Example 1: Use combinations to find the number of factors of 360. The prime factorization for 360 is $2^3 \times 3^2 \times 5$, so if we let the exponents be a, b, and c, it becomes $2^a \times 3^b \times 5^c$. This helps us visualize the possibilities. It is important to remember that any number written with an exponent (power) of 0 is equal to 1.

Example 2: Find the prime factorization of 84. How many possible factors are there of 84? List them all.

Section 5.4 Perfect Numbers

Looking Back 5.4

Remember that 2^3 is equal to $2 \times 2 \times 2$, so 2^3 is equal to 8; 2^n means multiply the number 2 by itself n times. If n is equal to 4, then 2 is multiplied by itself four times: $2^4 = 2 \times 2 \times 2 \times 2 = 32$. If n is equal to 5, then 2 is multiplied by itself five times: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 64$.

Many early mathematicians believed $2^n - 1$ is prime if n is prime. For example, if n is equal to 3, then $2^3 - 1$ is equal to $8 - 1$, which is equal to 7; in this instance, the early mathematicians would be correct. However, much confusion followed when in 1536, Hudalrichus Regius showed that $2^{11} - 1$ is equal to 2,047 and 2,047 is not prime. Regius did not even have a calculator; imagine how much work his calculations took!

In 1603, Pietro Cataldi found some numbers he thought were prime for different values of n. In 1647, fellow mathematician Pierre de Fermat proved that Cataldi was wrong about one of the numbers. In 1738, another mathematician, Leonhard Euler, proved that Cataldi was wrong about another one of the numbers.

Finally, a French monk named Marin Mersenne stated that $2^n - 1$ is a prime number when n is equal to 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, and 257. However, he was also wrong, but he did get his name attached to a group of numbers called Mersenne primes. By now, I think you get the picture: mathematicians often correct mathematicians! It is somewhat a competition amongst mathematicians to prove one another wrong.

Now enter what we call perfect numbers. A positive integer is perfect if it is equal to the sum of its factors, not including the number itself. Euler showed that there are an infinite number of prime numbers, but one of the unsolved mysteries of mathematics is whether or not there are an infinite number of perfect numbers. Maybe you will be the one to solve this mystery and your name will be attached to a group of numbers like Marin Mersenne!

For now, we will be using proper divisors to find perfect numbers. Proper divisors are positive integers that divide into a number without leaving a remainder, but do not include the number itself. A positive integer is a perfect number if it is equal to the sum of its proper divisors.

Looking Ahead 5.4

The proper divisor of any prime number is only one.

Composite numbers have at least two proper divisors.

Example 1:	Is 22 a perfect number? Is 13 a perfect number?

Example 2:	Demonstrate that 6 is a perfect number.

Example 3:	Demonstrate whether or not 12 is a perfect number.

Example 4:	Can you find the next perfect number after 6?

Section 5.5 Amicable, Abundant, and Deficient Numbers

Looking Back 5.5

There is a lot to learn about prime numbers. If prime numbers are the building blocks of all numbers, then perhaps they are useful after all! In the previous section, we learned about numbers that are called perfect numbers and how they relate to prime numbers. In this section, we will be introduced to three other types of numbers: Amicable, Abundant, and Deficient numbers.

Looking Ahead 5.5

The factors of a number are also called proper divisors because they divide into a number without leaving a remainder. When we learned multiplication, we learned about fact families, which are groups of math facts or equations that are created using the same set of numbers. These can be useful in working with the types of numbers included in this section.

Amicable numbers are two numbers such that each is the sum of the proper divisors of the other. This does not include the number itself.

Example 1:	Are 284 and 220 amicable numbers?

Amicable numbers are also called friendly numbers because they are so compatible.

Deficient numbers are numbers for which the _______________ of its proper ____________________ is

____________________ ____________________ itself. The number itself is not included in the factors.

Example 2:	Is 8 a deficient number?

Deficient numbers are also called defective numbers because they do not have enough factors (excluding the start number) to add together to equal the numbers themselves.

Abundant numbers are numbers for which the _______________ of its ______________ ______________

is ____________________ ____________________ the number itself. The number is not included in the factors.

Example 3:	Is 12 an abundant number?

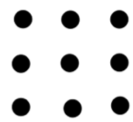
An ancient mathematician named Pythagoras (570-500 B.C.) was first to find the first pair of friendly numbers. Having a fascination with numbers and their origin, Pythagoras believed numbers themselves were the foundation for the universe. He did not believe in our God who created numbers.

Philosopher and mathematician René Descartes (1596-1650) found an incredible pair of friendly numbers: 9, 363,584 and 9, 437,056. In the 18th century, Leonhard Euler (who we have previously discussed) found a remarkable sixty pairs of friendly numbers. In 1866, Nicolo Paganini (a sixteen-year-old Italian) found one pair of numbers that had been overlooked: 1,184 and 1,210. This was all before computers and calculators were even invented!

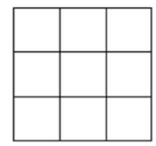
Section 5.6 Figurate Numbers

Looking Back 5.6

When a number is multiplied by itself, we say it is squared; we call these square numbers. The geometric representation of a square number is a square. A shape that is three dots long by three dots wide is a square of nine dots.



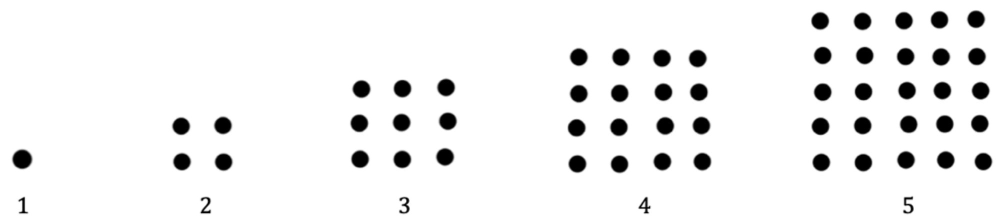
3 dots $\times$ 3 dots = 9 dots squared



3 square units by 3 square units = 9 units squared

Numbers that have geometric patterns like these are called figurate numbers.

Example 1:	Find the first five square numbers and the hundredth square number.



First square:

Second square:

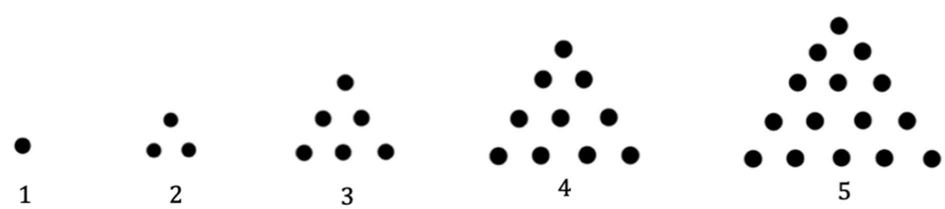
Third square:

Fourth square:

Fifth square:

Hundredth square:

Example 2:	Find the first five triangular numbers. Find the hundredth triangular number.



First triangle:

Second triangle:

Third triangle:

Fourth triangle:

Fifth triangle:

Hundredth triangle:

$$1 + 2 + 3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots .. +98 + 99 + 100$$

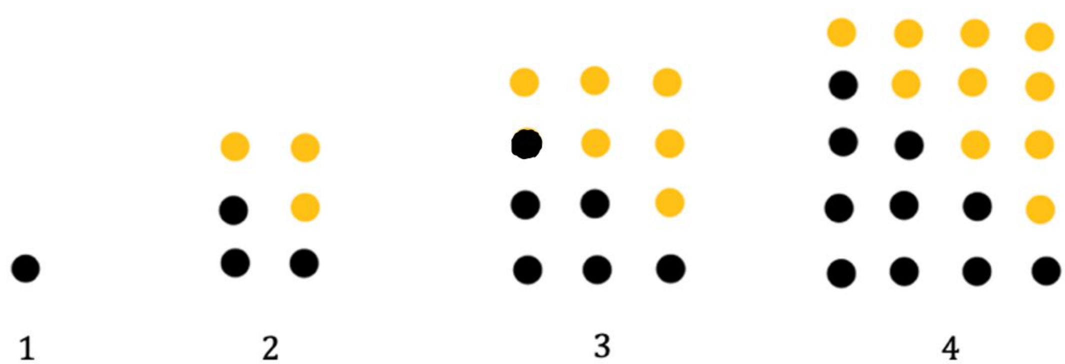
$$3 + 98 = 101$$

$$2 + 99 = 101$$

$$1 + 100 = 101$$

This was discovered by Karl Friedrich Gauss, a German mathematician. Gauss' teacher at school wanted to keep him busy mathematically and asked him to find the sum of the numbers 1 through 100. You can imagine her astonishment when Gauss used this method and instantly replied with the solution of 5,050! Gauss also went on to prove the Fundamental Theorem of Arithmetic, which states that every natural number can be represented as a product of primes.

Example 3: Use the area method to find the formula that Gauss used to find the sum of numbers 1 through 100.



Section 5.7 Fibonacci Numbers

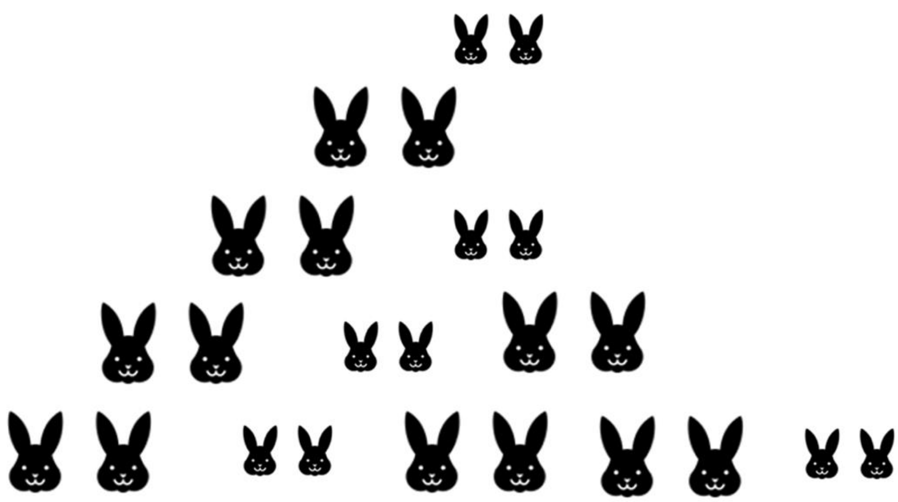
Looking Back 5.7

Leonardo de Pisa was a mathematician who wrote *The Book of Squares*. He also replaced the Roman numeral system with the Hindu-Arabic system, which we use today. De Pisa traveled to Egypt, Greece, Sicily, and Syria with his father, a merchant, gaining knowledge of mathematics and the world at a young age. Therefore, when the Emperor of Pisa, Frederick II, held a calculating competition (as he often did) to find who could best serve him as Head Mathematician for his domain, De Pisa solved the rabbit problem. This resulted in De Pisa receiving funds from the emperor to continue his work.

The rabbit problem is a famous problem that asks the question: "Beginning with a single pair of rabbits, if every month the productive pair bears a new pair and when the new pairs are one-month old, they bear another pair and every pair continues bearing a new pair every month, how many rabbits will there be in a year?" This classic problem leads up to the Fibonacci sequence, a fascinating constant in nature. We will look at three different examples beginning with the rabbit problem.

Looking Ahead 5.7

Example 1: "Beginning with a single pair of rabbits, if every month the productive pair bears a new pair and when the new pairs are one-month old, they bear another pair and every pair continues bearing a new pair every month, how many rabbits will there be in a year?"



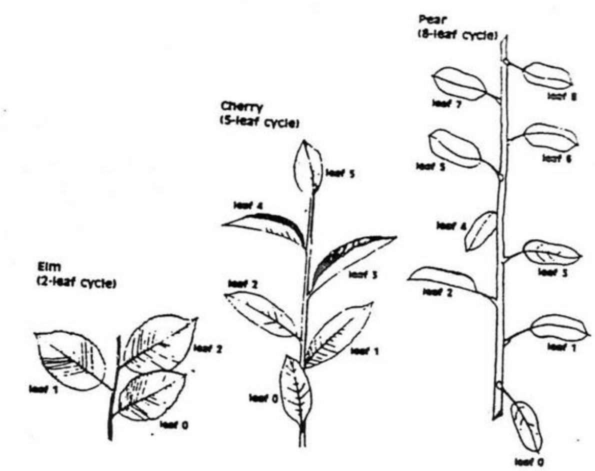
There are Fibonacci numbers everywhere in nature. It is not a surprise that botanists have found this pattern over and over; God created this sequence with design in mind as can be seen throughout nature!

The Fibonacci sequence is shown below:

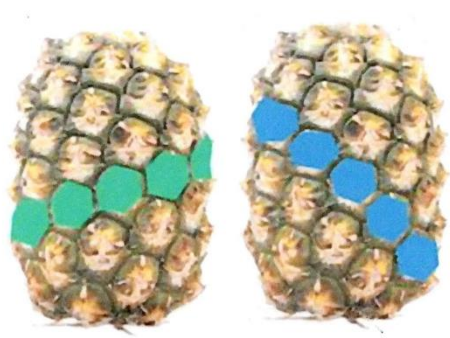
$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$ ...

After the first two digits, each number in the sequence is the ____________ of the previous ____________ numbers.

Example 2: Phyllotaxis is the arrangement of leaves on a stem. Going from one leaf to the next similarly placed leaf is one complete turn (revolution). Count the leaves in one turn (revolution) on each plant shown below. Are they all Fibonacci numbers?



Example 3: Pineapples are formed from hexagons. There are three distinct hexagon spirals that occur in nature: a first group of five winding in one direction; a second group of thirteen winding steeply in the same direction; a third group of eight winding in the opposite direction. Count the spirals highlighted below. What type of number do you find?



The seeds of a sunflower and the seeds of an artichoke are arranged in spirals of Fibonacci numbers. The seeds in the center of a daisy and sunflower are arranged in two intersecting spirals and each is a set of Fibonacci numbers; one goes clockwise and the other goes counterclockwise. The scales on a pinecone are spirals of Fibonacci numbers. Fibonacci numbers are found throughout God's natural world. Get a pineapple or pinecone this week and count the Fibonacci numbers.

Section 5.8 Fibonacci Numbers and the Golden Ratio

Looking Back 5.8

As we have learned, a ratio is a comparison of two numbers. We usually leave a ratio as a fraction but they can be divided to find a quotient. A very special quotient known as the Golden Ratio is often found in architecture. The Golden Ratio produces proportions that are very pleasing to the eye. The Greeks used this ratio to create statues of human likeness that represent perfect figures. They also built the Parthenon using the Golden Ratio. In this section, we are going to do some activities to help us become familiar with this beautiful number and its exquisite ratio.

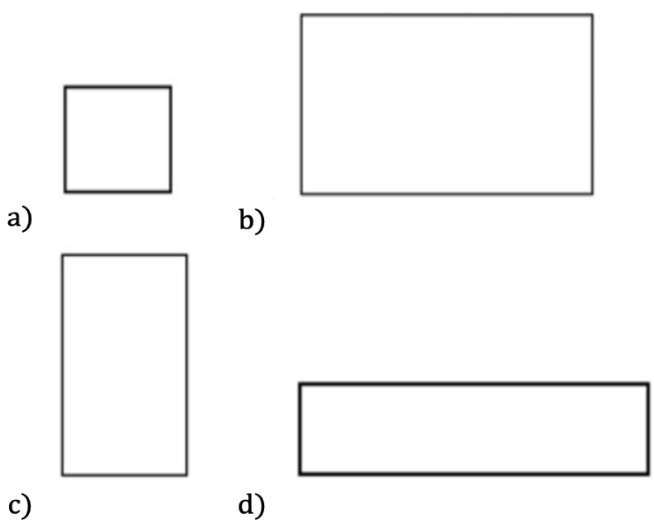
Looking Ahead 5.8

Activity 1: Measure the length and width of a single playing card from a deck of cards and find the ratio of its length to width.

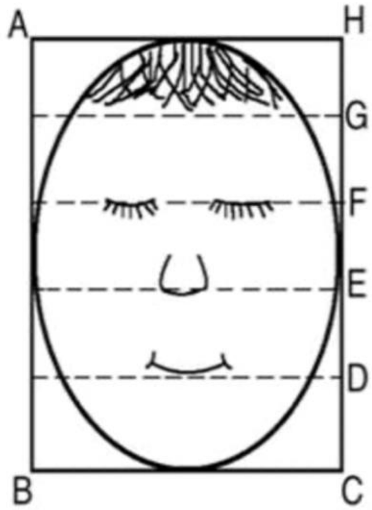
$$\frac{\text{Length of Playing Card}}{\text{Width of Playing Card}} = \frac{\_\_\_\_\_\_\_\_\_\text{ cm.}}{\_\_\_\_\_\_\_\_\_\text{ cm.}}$$

What number do you get when you divide the length by the width?

Activity 2: Pick the rectangle below that you like the best.



Activity 3: Use a metric ruler (in millimeters) to measure these parts of the head and divide the necessary fractions (shown below) to find the decimal approximation that decides whether a person is the golden ratio or not.



$\frac{HC}{BC} =$

$\frac{HF}{FE} =$

$\frac{FC}{EC} =$

Are you golden?!

Activity 4: Measure the parts of your body described below and find your ratios.

$\frac{\text{Shoulder to Fingertip}}{\text{Elbow to Fingertip}} =$ $\frac{\text{Head to Toe (Height)}}{\text{Waist to Floor}} =$ $\frac{\text{Thigh to Floor}}{\text{Knee to Floor}} =$

By now, you have discovered whether you are the Golden Ratio or not, but either way, you are golden. Proverbs 139 says that you are "fearfully and wonderfully made." God created you in His image and you are pleasing in His sight. This is why playing cards are the Golden Ratio: to fit in your golden hands! Any Fibonacci number divided by the previous number in the sequence approaches the Golden Ratio.

The Golden Spiral, which we will learn how to make in the Practice Problems section, is a logarithmic spiral with a growth factor that is the Golden Ratio! It can be found in nature as well: in seashells, the horns of rams, the tails of seahorses, and many other places. Go outside and look for the Golden Spiral!

Section 5.9 Adding with Exponents

Looking Back 5.9

Square numbers are any base to the second power. We have learned a little bit more about exponents when we worked with perfect square numbers. Let us review them now.

Read: "x to the second power."	Read: "six to the second power."
x^2	6^2
$x \cdot x$	$6 \cdot 6$
This means we have two xs multiplied together	This means we have two 6s multiplied together
base $\longrightarrow x^2 \longleftarrow$ exponent	base $\longrightarrow 6^2 \longleftarrow$ exponent

The base is the variable or constant (or both) that is (are) being multiplied a certain number of times. The exponent, sometimes called the "power," tells us how many times to multiply the base.

Looking Ahead 5.9

Example 1: If the bases are different and the exponents are different for the sum of two or more terms, can we add bases to bases and exponents to exponents to get the correct answer?

$$2^3 + 3^2$$

Example 2: If the bases are the same but the exponents are different for the sum of two or more terms, can we keep the bases the same and add the exponents?

$$2^3 + 2^2$$

Example 3: If the bases are different but the exponents are the same for the sum of two or more terms, can we add the bases and keep the exponents the same?

$$2^3 + 3^3$$

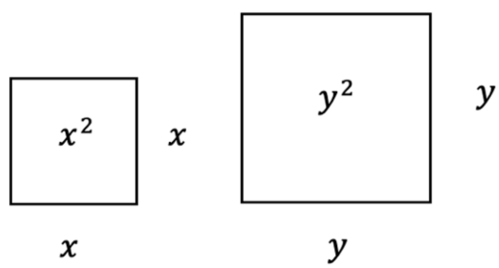
Example 4: If the bases are the same and the exponents are the same for the sum of two or more terms, can we add the bases and keep the exponents the same?

$$2^3 + 2^3$$

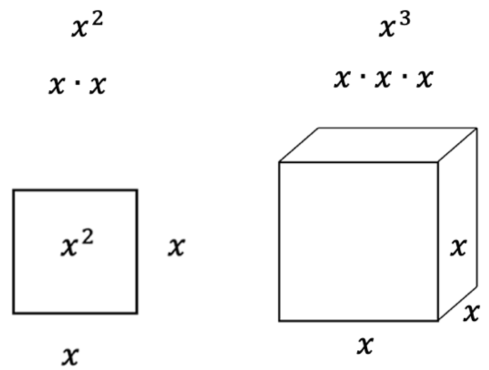
Example 5: If the bases are the same and the exponents are the same for the sum of two or more terms, can we keep the bases the same and add the exponents?

$$2^3 + 2^3$$

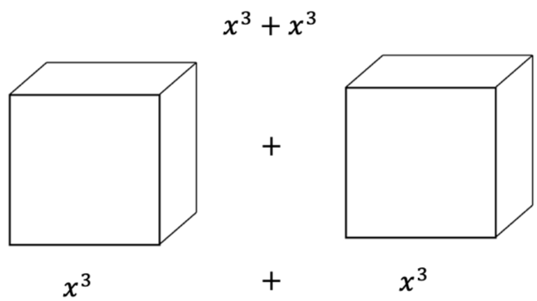
Example 6: Can $x^2 + y^2$ be simplified in terms of all x or all $y's$?



Example 7: What if the variables have the same base but different exponents for two or more terms? For example, in $x^2 + x^3$ can the terms be combined?



Example 8: What if the variable bases are the same and the exponents are the same for two or more terms? Can the terms be combined?



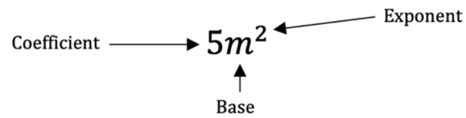
Like terms are variables with the same bases and the same exponents. When we add like terms, we call it "combining like terms."

Now, we have a rule for combining like terms: If the bases are the same and the exponents are the same, combine the like terms. When combining like terms that are being added, we add the coefficients of the like terms; we keep the common base and the common exponent the same.

Section 5.10 Subtracting with Exponents

Looking Back 5.10

Addition and subtraction are inverse operations. The same rules that apply to addition involving exponents of variables, constants, or both, apply to subtraction involving exponents of variables, constants, or both. Remember, $5m^2$ is called a monomial term:



A monomial is a number, variable, or the multiplication of a number and a variable. The monomial here is $5m$; a monomial can be 5, m, or $5m$.

The coefficient is a number that is in front of the variable; it does not change.

The base is a variable if there is a coefficient. It varies (can be different) from problem to problem. The exponent here is 2 and will stay the same because it is a number. Monomial terms are separated by plus or minus signs.

Looking Ahead 5.10

Example 1:	Subtract the monomial terms with exponents below.

$$2^5 - 3^3$$

Example 2:	Subtract the monomial terms below.

$$8n - 5n$$

Example 3:	Combine the like terms in the expression below.

$$13t - 111t + 5t - 7t$$

Section 5.11 Multiplying with Exponents

Looking Back 5.11

Three raised to the second power means 3×3, so b raised to the n^{th} power means $b \times b \times b$ ... etc. n times.

$$b^n = \underbrace{b \times b \times b \times \ldots \times b}_{n \text{ factors}}$$

What happens when we multiply monomial terms with exponents; for example, $b^n \times b^n$? Simple mathematical concepts provide the foundation for difficult mathematical concepts; let us start with what we know.

We know that $x \cdot x$ is x^2 because x is being multiplied by itself two times. We know that $x \cdot x \cdot x$ is x^3 because x is being multiplied by itself three times.

Looking Ahead 5.11

Example 1:	Is $2^2 \times 2^3$ equal to $2^{(2+3)}$ or $2^{(2\text{x}3)}$?

When multiplying exponents with ___________________ bases, ___________________ the exponents.

Example 2:	Simplify the expression below.

$$n^5 \times n^7$$

Example 3:	Simplify the expression below.

$$5m^3 \cdot -3m^7$$

Because the bases are the same, we still __________________ the exponents, but we _______________________ the coefficients because it is a multiplication problem.

Example 4:	Simplify the expression below.

$$3m^3(-6n^4)$$

Example 5:	Simplify the expression below.

$$-2x^2y(-6xy^2)$$

Section 5.12 Dividing with Exponents

Looking Back 5.12

Division is the inverse operation of multiplication. If we add exponents when we multiply with like bases, what operation will we use with the exponents when we divide with like bases? Let us try it with real numbers and see what happens.

Looking Ahead 5.12

Example 1:	Simplify the expression below.

$$\frac{3^6}{3^4}$$

Did we add the exponents like we did with multiplication?

Example 2:	Simplify the expression below.

$$\frac{y^8}{y^4}$$

Example 3:	Simplify the expression below.

$$\frac{m^5}{m^9}$$

What happens when the exponent is negative?

To divide with like bases, ____________________ the exponents.

$\frac{b^n}{b^m} = b^{n-m}$ (For every non-zero number)

For now, let us further investigate what happens in operations with negative exponents.

The Transitive Property and Substitution Property

The Transitive Property and Substitution Property are very similar. The Substitution Property is used when we want to substitute a number in for a variable in an equation. For example, if $x + 5 = 21$ and $x = 16$, then we substitute 16 in for x and then we have the equation $16 + 5 = 21$. If $x + 5 = y$ and $y = d$, then $x + 5 = d$; in this case, we substituted d in for y.

The Transitive Property is like a special case of substitution, but it involves substituting in values that are equal to other values. For example, if $x = y$ and $y = z$, then $x = z$. In this case, think of it more as a "chain reaction." Let us suppose $1.5 = \frac{3}{2}$ and $\frac{3}{2} = 1\frac{1}{2}$, then $1.5 = 1\frac{1}{2}$.

Example 4:	Simplify the expression below.

$$\frac{n^9}{n^9}$$

The rule for negative exponents is the following:

$\frac{1}{b^n} = b^{-n}$ and $b^{-n} = \frac{1}{b^n}$
(In which $b \neq 0$)

Example 5:	Using the rule for negative exponents, rewrite the expression below so it is not a fraction.

$$\frac{1}{5^2}$$

Example 6:	Simplify the expression below.

$$\frac{15x^9}{3x^3}$$

Example 7:	Simplify the expression below.

$$\frac{36x^2}{9x^7}$$

When dividing powers with like bases, subtract the exponents and divide the coefficients because it is a division problem.

Section 5.13 Power of a Power

Looking Back 5.13

We know that when dividing with like bases and exponents that are equal, $\frac{b^n}{b^n} = 1$ because $b^{n-n} = b^0 = 1$. We also know that when we multiply exponents with like bases, we add the exponents. We will complete our lessons on exponents by learning about the rule for the power of a power (also known as the power of a product rule).

Looking Ahead 5.13

When a product is taken to a power, each term is taken to the exponent and $(ab)^n$ is equal to $a^n b^n$ for every nonzero number a and b and every integer n. For example, $(3x)^2 = 3^2x^2 = 9x^2$. This is because $(3x)^2 = (3x)(3x) = 9x^2$; the coefficient and variable is multiplied by itself. Because multiplication is commutative, the order of the numbers and variables can be arranged without affecting the answer.

Example 1:	Simplify the expression below.

$$(4y)^3$$

Example 2:	Simplify the expression below.

$$(2x^2)^4$$

Example 3:	Simplify the expression below.

$$(5x^4)^3$$

For every nonzero number b and integer m and n, $(b^m)^n = b^{mn}$. To take a power to a power, multiply the exponents.

Example 4:	Simplify the expression below.

$$(y^5)^4$$

Example 5:	Simplify the expression below.

$$(2^3)^2$$

Example 6:	Simplify the expression below.

$$-(x)^4$$

If there is no exponent, the exponent is equal to one. If there is no coefficient, the coefficient is equal to one.