Algebra 2 Module 5 Roots and Radicals

Section 5.1 Roots of Real Numbers

Looking Back 5.1

The square root of a number "a" is a solution of the equation $x^2 = a$. If a is positive there are two solutions, $+\sqrt{a}$ and $-\sqrt{a}$. The positive square root is called the principal square root.

$$1 \cdot 1 = 1 \text{ and } \sqrt{1} = +1 \text{ or } -1$$

$$2 \cdot 2 = 4 \text{ and } \sqrt{4} = +2 \text{ or } -2$$

$$3 \cdot 3 = 9 \text{ and } \sqrt{9} = +3 \text{ or } -3$$

$$x \cdot x = x^2 \text{ and } \sqrt{x^2} = +x \text{ or } -x$$

The square of a real number is always positive so $x^2 = a$ has no real number solution if $a < 0$. If a is a negative number, there are no real square roots. There are, however, non-real solutions, which you learned about in the previous module. You will study these again later in this module.

Example 1:	Simplify the radicals.

a) $\sqrt{16}$

b) $-\sqrt{25}$

c) $\sqrt{\frac{1}{36}}$

d) $\sqrt{0.49}$

Example 2:	Find the real roots of each equation.

a) $x^2 = 81$

b) $x^2 - 4 = 0$

c) $10x^2 = 200$

The cube root of "a" is a solution to the equation $x^3 = a$. There is only one solution whether "a" is negative, zero, or positive.

Example 3:	Find the real roots of each equation.

a) $x^3 = 27$

b) $x^3 = 10^9$

c) $\sqrt[3]{x^3}$

<u>Looking Ahead 5.1</u>

Properties of Radicals

1. A solution to $x^n = a$ is $\sqrt[n]{a}$. Therefore, $\left(\sqrt[n]{a}\right)^n = a$.

Examples:

$$\left(\sqrt{8}\right)^2 = 8$$

$$\left(\sqrt[3]{-2}\right)^3 = -2$$

2. If n is even, $\sqrt[n]{a^n} = |a|$. The principal square root is a non-negative number when n is even.

Examples:

$$\sqrt{(-4)^2} = |-4| = 4$$

$$\sqrt{(x-2)^2} = |x-2| = \begin{cases} x-2 \text{ if } x \geq 2 \\ 2-x \text{ if } x < 2 \end{cases}$$

3. If n is odd, $\sqrt[n]{a^n} = a$.

Examples:

$$\sqrt[3]{5^3} = 5$$

$$\sqrt[7]{x^7} = x$$

Example 4:	Tell whether the equation is true or false for all real numbers.

a) $\sqrt{y^4} = y^2$

b) $\sqrt[7]{x^7} = |x|$

c) $\left(\sqrt[7]{x}\right)^7 = |x|$

In summary:

For radicals with a positive integer index:

1. $(\sqrt[n]{a})^n = a$ if $\sqrt[n]{a}$ is a real number (Note that $(\sqrt[2]{-3})^2$ is not a real number because $\sqrt[2]{-3}$ is not a real number)

2. $\sqrt[n]{a^n} = a$ if $a \geq 0$ (Example: $\sqrt{4^2} = 4$; Example: $(\sqrt[3]{(3)^3} = 3)$

3. $\sqrt[n]{a^n} = |a|$ if $a < 0$ and n is even (Example: $\sqrt[4]{(-2)^4} = |2| = 2)$

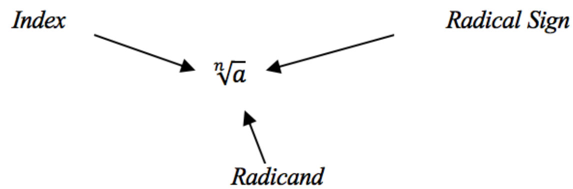
4. $\sqrt[n]{a^n} = a$ if $a < 0$ and n odd (Example: $\sqrt[3]{(-2)^3} = -2$

Section 5.2 Roots of Non-Real Numbers

Looking Back 5.2

When the equation is $x^2 = 5$, as in the previous practice problems section, then $x^2 = 5$, $\sqrt{x^2} = \pm\sqrt{5}$, $x = \pm\sqrt{5}$ and there are two exact solutions. The number 5 is a prime number and has no factors that are perfect squares. It can be left in exact form. There is also a decimal approximate, $x = \pm\sqrt{5} \approx 2.23607$, which could be rounded to 2.2, 2.24, or even 2.236. We will discuss these irrational numbers later in the module.

The number under the radical sign, the radicand, whose index is two, may have a factor that is a perfect square and another factor whose square root is irrational. The perfect square can be found and the factor whose square root is irrational can be left under the radical sign.



Example 1:	Simplify the radicals and find the exact answer.

a) $\sqrt{41}$

b) $\sqrt[3]{81}$

c) $\sqrt{40x^2y^3}$

d) $\sqrt[5]{x^5y^{10}z^7}$

Note: The solution to c) may be written $2|x||y|\sqrt{10y}$ and the solution to d) may be written $|x| \cdot y^2 \cdot |z| \cdot \sqrt[5]{z^2}$.

Looking Ahead 5.2

In the last module, you learned about imaginary numbers, which were numbers we used when the determinant of the quadratic formula was negative.

The imaginary number i is used to simplify square roots of negative numbers.

Definition of i:

$$i^2 = -1 \qquad \sqrt{i^2} = \pm\sqrt{-1} \qquad i = \sqrt{-1}$$

$$(2i)^2 = 2^2 \cdot i^2 = 4i^2 = 4(-1) = -4$$

If $(2i)^2 = -4$, then $\sqrt{(2i)^2} = \pm\sqrt{-4}$

$$2i = \pm\sqrt{-4}$$

The principle square root is $2i = \sqrt{-4}$.

$$2i = 2\sqrt{i^2}$$

$$2i = 2i$$

$$(3i)^2 = 3^2 \cdot i^2 = 9i^2 = 9(-1) = -9$$

$$(3i)^2 = -9$$

$$\sqrt{(3i)^2} = \pm\sqrt{-9}$$

$$3i = \pm\sqrt{-9}$$

The principal square root is $3i = \sqrt{-9}$.

$$3i = 3\sqrt{i^2}$$

$$3i = 3i$$

Example 2:	Simplify the following radicals and find the non-real solutions.

a) $\sqrt{-5}$

b) $\sqrt{-20}$

c) $\sqrt{-13x^2y^4z}$

d) $\sqrt{-64}$

The solution to c) may be written $|x| \cdot y^2 \cdot i\sqrt{13z}$.

Example 3:	Simplify the fractions.

a) $\frac{2}{5i}$

b) $\frac{6}{\sqrt{-5}}$

Section 5.3 Radicals and nth Roots

Looking Back 5.3

An nth root of a is a solution of $x^n = a$.

1. If n is even and $a > 0$, there are two real nth roots of a. The positive (principal) nth root of a is $\sqrt[n]{a}$. The other nth root of a is $-\sqrt[n]{a}$.

2. If n is odd there is exactly one nth root of a. This is true when a is negative, zero, or positive.

3. If n is even or odd there is one nth root when $a = 0$, written $\sqrt[n]{0} = 0$.

We reviewed those properties in Section 1 of this module. In Section 2, the previous section, we reviewed Property 4:

4. If n is even and $a < 0$, there is no real nth root of a.

Example 1:	Show that 5 and -5 are both fourth roots of 625.

Looking Ahead 5.3

In x^n, x is the base and n is the exponent, but x^n is called a power function. The nth root of a non-negative number can be expressed as a power, where $\frac{1}{n}$ is the exponent.

The power of a power rule states that $(x^m)^n = x^{mn}$. An example is $(27^{\frac{1}{3}})^3 = 27^{\frac{3}{3}} = 27^1 = 27$. This is because $27^{\frac{1}{3}}$ is the cube root of 27; $\sqrt[3]{27} = 3$ and $3^3 = 27$, and $(\sqrt[3]{27})^3 = 27$. Therefore, $\sqrt[n]{x} = x^{\frac{1}{n}}$.

If $x \geq 0$ and $n \geq 2$ (an integer only), then $\sqrt[n]{x} = x^{\frac{1}{n}}$ is the positive nth root of x.

If $x < 0$ and n is an odd integer, then $\sqrt[n]{x} =$ the real nth root of x.

If $x < 0$, then $x^{\frac{1}{n}}$ is not used.

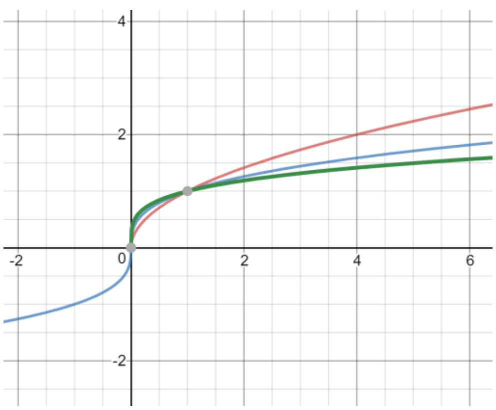
If $x < 0$ and n is even, the $\sqrt[n]{x}$ is not a real number.

Example 2:	Rewrite $16{,}384^{\frac{1}{4}}$ as a radical and simplify.

Use the $\sqrt[n]{x}$ key on the calculator (it should be above the ∧ key. Press "ctrl ∧" to get the command above it (the second command).

Example 3: The equations $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$ are inverses. Demonstrate how they are inverses using $f(g(x))$ and $g(f(x))$.

Example 4: The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown below. Sketch the graphs of $y = x^{\frac{1}{5}}$, $y = x^{\frac{1}{6}}$ and $y = x^{\frac{1}{7}}$.



Section 5.4 Rational Exponents

Looking Back 5.4

There are properties of powers that work for all real numbers.

For a product of powers:

$$x^m \cdot x^n = x^{m+n}$$

For a power of a product:

$(xy)^m = x^m y^m$ or $(xy)^n = x^n y^n$

$$(x^m y^n)^p = x^{mp} y^{np}$$

For a quotient of a power

$$\frac{x^m}{x^n} = x^{m-n}$$

For a power of a quotient:

$(\frac{x}{y})^m = \frac{x^m}{y^m}$ or $(\frac{x}{y})^n = \frac{x^n}{y^n}$

Any base to the zero power equals 1:

$a^0 = 1$ $\qquad \frac{a^2}{a^2} = a^{2-2} = a^0 = 1$

Any base to a negative exponent is the reciprocal of the base with a positive exponent:

$x > 0$ and $x \neq 0$ (n is an integer)

$x^{-n} = \frac{1}{x^n}$ $\qquad \frac{1}{x^2} = \frac{x^0}{x^2} = x^{0-2} = x^{-2}$

Example 1:	Simplify the following numbers.

a) $2^4 \cdot 2^{-4}$

b) 4^{-2}

c) $117{,}649^{\frac{5}{6}}$

d) $27^{-\frac{5}{3}}$

For all integers *m* and *n* and *x* > 0 then $x^{m/n} = (x^{\frac{1}{n}})^m = (x^m)^{\frac{1}{n}}$ and $(\sqrt[n]{x})^m = \sqrt[n]{x^m}$.

Looking Ahead 5.4

Example 2:	Simplify $64^{\frac{4}{3}}$.

Example 3:	Show that $x^{\frac{2}{7}}$ is $\sqrt[7]{x^2}$ using the properties of exponents.

Section 5.5 Operations with Radicals
Looking Back 5.5

As long as the radicands are the same, radicals can be added and subtracted. The coefficients of the common radicands are added or subtracted.

Example 1:	Simplify the radicals by combining like terms.

$$\sqrt{4} + 2\sqrt{4} - 3\sqrt{4}$$

Example 2:	Simplify the radicals by combining like terms.

$$5\sqrt{3} + \sqrt{4} + \sqrt{5} - 3\sqrt{3}$$

It is not correct to separate the radical over the addition or subtraction sign.

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$$

For example, $\sqrt[2]{25+9} = \sqrt[2]{34}$; however, $\sqrt[2]{25+9} \neq \sqrt[2]{25} + \sqrt[2]{9}$.

This is because $5 + 3 = 8$, not $\sqrt[2]{34}$.

$$\sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$$

For example, $\sqrt[2]{25-9} = \sqrt[2]{16} = 4$; however, $\sqrt[2]{25-9} \neq \sqrt[2]{25} - \sqrt[2]{9}$.

This is because $5 - 3 = 2$, not 4.

Looking Ahead 5.5

However, for all real numbers a and b, it is correct to separate the radical over the multiplication or division sign.

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\sqrt[3]{8 \cdot 27} = \sqrt[3]{8} \cdot \sqrt[3]{27}$$

$$\sqrt[3]{216} = 2 \cdot 3$$

$$6 = 6$$

And…

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$$

$$\frac{2}{3} = \frac{2}{3}$$

Notice that the index of both is 3. They must be the same. If the indexes are different, they cannot be multiplied or divided.

Also, $\sqrt[3]{\frac{8}{27}}$ means take the cube root of the numerator and the cube root of the denominator. If you divide 8 by 27 and then take the cube root, you will get the decimal approximation $\sqrt[3]{\frac{8}{27}} \approx \sqrt[3]{0.296\ldots} \approx 0.6667\ldots$ instead of the exact answer.

Remember to simplify the denominator if it has a radical in it. Rationalize the denominator to eliminate the radical.

Example 4:	Simplify the radicals and rationalize the denominator.

$$\sqrt[3]{\frac{216}{4}}$$

Section 5.6 Products of Binomials with Radicals

Looking Back 5.6

With radicals the base is called the radicand. Radicals of any radicand may be multiplied, but the index must be the same.

Just as x and y give a product xy, $\sqrt{x} \cdot \sqrt{y} = \sqrt{xy}$.

Example 1:	Simplify the radicals.

$$3x(2x\sqrt{3})$$

Example 2:	Use the distributive property to simplify the radicals.

$$3\sqrt{3}(2 + 5\sqrt{5})$$

Looking Ahead 5.6

You multiply binomials containing radicals the same way you multiply binomials with real numbers and variables.

Example 3:	Multiply the binomials using the distributive property.

$$(4 + 3\sqrt{2})(2 - 2\sqrt{5})$$

Example 4:	Use long multiplication to multiply the binomials.

$$(4 + \sqrt{6})(3 - \sqrt{6})$$

Example 5:	Check the problem in Example 4 by multiplying using the distributive property.

$$(4+\sqrt{6})(3-\sqrt{6})$$

Remember the special pattern $(a-b)^2 = a^2 - 2ab + b^2$. The difference of a binomial square gives a perfect trinomial square when expanded using multiplication.

Example 6:	Expand $(2\sqrt{5}-\sqrt{3})^2$. Check the answer using the special pattern above.

Section 5.7 Quotients of Binomials with Radicals

Looking Back 5.7

Just as variables with negative exponents are not left in the denominator when expressions with exponents are simplified, radicals are not left in the denominator when radicals are simplified. In order to simplify radicals that are quotients the denominator must be rationalized. The denominator must first be rationalized by a unit of one, which will eliminate the radical in the denominator. Numerous examples of radical units of one are listed as follows:

$$\frac{\sqrt{3}}{\sqrt{3}} \text{ or } \frac{\sqrt{x}}{\sqrt{x}} \text{ or } \frac{3\sqrt{x}}{3\sqrt{x}} \text{ or } \frac{x\sqrt{3}}{x\sqrt{3}}$$

There are infinitely many more!

Example 1:	Simplify $\frac{\sqrt{5}-\sqrt{12}}{\sqrt{2}}$.

Example 2:	Simplify $\sqrt{\frac{2}{3}}\left(\sqrt{\frac{3}{2}}-\frac{3}{\sqrt{2}}\right)$.

Example 3: Simplify $\sqrt[3]{1}-\sqrt[3]{\frac{1}{3}}$.

Looking Ahead 5.7

Radicals of any radicand may be multiplied, but the index must be the same.

You learned about conjugates in the prior module; $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are conjugates because when multiplied together they give a real number. The conjugate of $c\sqrt{a}+d\sqrt{b}$ is $c\sqrt{a}-d\sqrt{b}$. They are exactly the same besides the sign. This makes the first and last term perfect squares and then the middle term is eliminated because they become additive inverses, which equal 0.

Example 4: Multiply $\left(3+\sqrt{5}\right)$ by $(3-\sqrt{5})$.

Example 5: Divide $3+\sqrt{5}$ by $3-\sqrt{5}$.

Section 5.8 Solving Equations Containing Radicals

Looking Back 5.8

An equation that has a radical with a variable in the radicand is called a radical equation. To solve a radical equation with a square root, take the square of both sides of the equation. To solve an equation with a cube root, take the cube of both sides of the equation. To solve a radical equation with an nth root, multiply by the nth power on both sides of the equation.

Example 1: Solve $\sqrt{3x+1} = 4$.

Example 2: Solve $5\sqrt[3]{x} - 2 = 3$.

Looking Ahead 5.8

Sometimes, when the radical is transformed algebraically, one of the roots that is a solution of the transformation is not a root of the equation in its original form. This solution is called extraneous.

Example 3:	Solve $3y = 2 + 5\sqrt{y}$.

Example 4:	Solve the equation without squaring each side.

$$2x = 3 + x\sqrt{7}$$

Section 5.9 Graphs of Radicals

Looking Back 5.9

Roots (radicals) and exponents (powers) are inverses.

x^2 and $\pm\sqrt{x}$

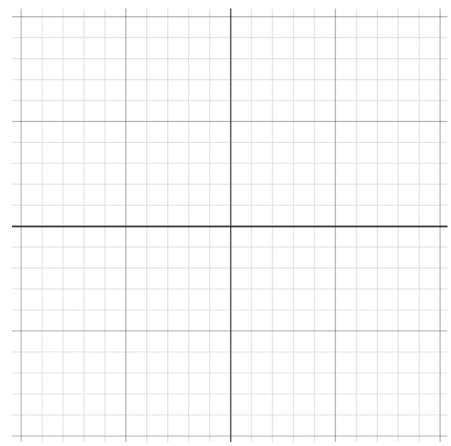
x^3 and $\sqrt[3]{x}$

Firstly, we will review those graphs.

Complete the tables and graph the above functions:

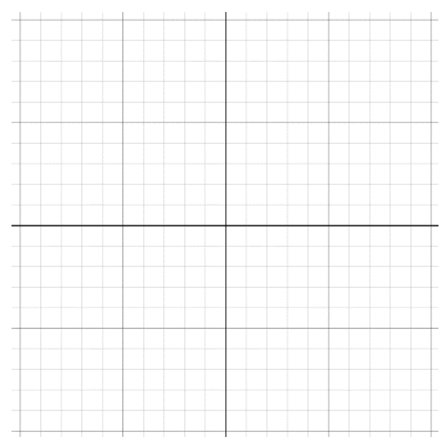
1. $y = x^2$

x	y
-2	
-1	
0	
1	
2	



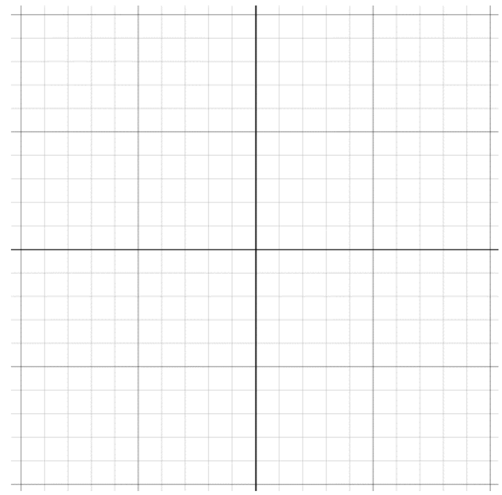
2. $y = \pm\sqrt{x}$

x	y
4	
1	
0	
1	
4	



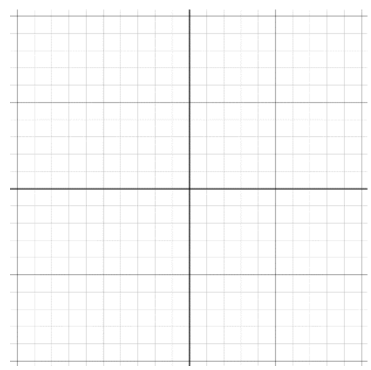
3. $y = x^3$

x	y
-2	
-1	
0	
1	
2	



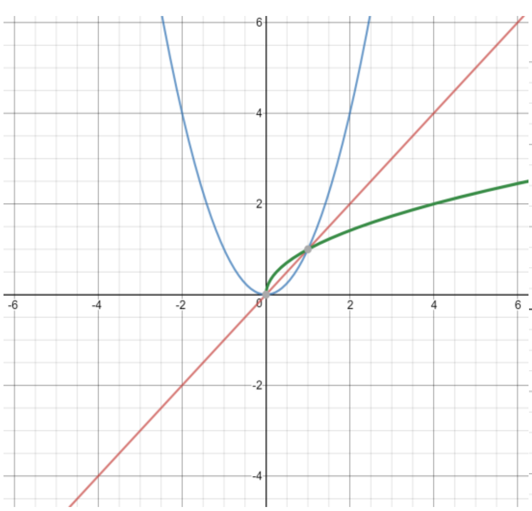
4. $y = \sqrt[3]{x}$

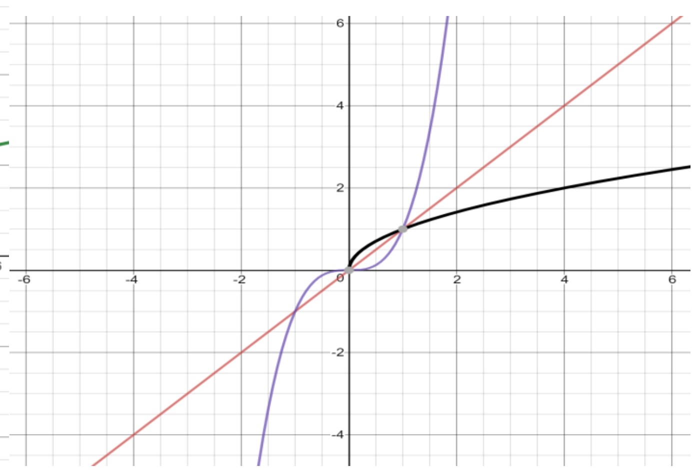
x	y
-8	
-1	
0	
1	
8	



Looking Ahead 5.9

We have been investigating the principal square roots of radicals. The inverse of $f(x) = x^2$ is $f(x) = \pm\sqrt[2]{x}$. All power equations are functions. Are their inverses functions also?





Section 5.10 Rational and Irrational Numbers

Looking Back 5.10

In the previous sections the answers have been written as exact answers or sometimes, as decimal approximations.

Example 1:	Find the exact answer to the linear equation and check your answer.

$$3x + 2 = x\sqrt{5}$$

What if I asked you to find the decimal approximation and check your answers? Is it possible?

$$\frac{-\sqrt{5} - 3}{2} \approx -2.61803$$

This number is irrational. It does not repeat and does not end. It would be impossible to check unless you are the Almighty Mathematician, and there is only one of them, the Almighty God.

Any radical that is not a perfect square or a perfect cube is not a rational number. Rational numbers repeat or end and can be written as the quotient of two numbers, $\frac{a}{b}$, and $b \neq 0$. The variables a and b are integers. If $a = 0$, the solution is 0.

Unless we are solving a real-world problem, we use exact answers for radical equations.

Looking Ahead 5.10

Have you ever wondered how 0.333… can be written as $\frac{1}{3}$? Because it repeats, it is a rational number; because it follows a pattern it makes sense.

Example 2:	Write the repeating decimal as a ratio. Simplify the common fraction to lowest terms.

$$0.3333 ... = 0.\overline{3}$$

$$\text{Let } N = 0.\overline{3}$$

Let n be the number of places of the repeating part of the decimal. In this case, $n = 1$ because only one number repeats, 3. Multiply by 10^n and solve the system of equations.

Example 3: Tell whether the numbers are rational or irrational.

a) $\sqrt{5}$

b) $\sqrt{\frac{9}{36}}$

Example 4: $\frac{3}{7} \approx 0.428571428571\ ...$

Show that there is a block of six numbers that repeat.

Section 5.11 Complex Numbers

Looking Back 5.11

Let's review the definition of imaginary numbers, which you learned in the previous module for non-real solutions of quadratic equations involving complex numbers.

$$i = \sqrt{-1} \text{ and } i^2 = -1$$

$$\text{If } x^2 = -9\text{, then } \sqrt{x^2} = \pm\sqrt{9 \cdot -1}$$

$$x = \pm 3i$$

Both $3i$ and $-3i$ are solutions of $x^2 = -9$

$$(3i)^2 = 3^2 i^2 = 9i^2 = 9(-1) = -9$$

$$(-3i)^2 = (-3)^2 i^2 = 9i^2 = 9(-1) = -9$$

$$3i = \sqrt{-9}$$

A pure imaginary number is of the form $a + bi$ where $a = 0$. Since the real part is 0, only the imaginary part is represented.

Example 1: Simplify.

a) $\sqrt{-11}$

b) $\sqrt{-49}$

c) $\sqrt{-90}$

Example 2: Simplify.

a) $\sqrt{-9x^2}$

b) $\sqrt{-x^4}$

c) $\sqrt{-50y^3}$

Looking Ahead 5.11

You have had a few opportunities to learn, practice, and review these concepts. Now that you are familiar with them, we are going to spend these next few sections putting all that you have learned together before you move on to the more complicated radicals.

Example 3: Solve for *x*. Find the non-real solutions.

$$x^2 + 169 = 0$$

Example 4: Solve for *x*. Find the non-real solutions.

$$4x^2 + 3 = -49$$

Example 5: Simplify the radical to its pure imaginary form.

$$\frac{\sqrt{50}}{\sqrt{-10}}$$

Section 5.12 Sums of Complex Numbers

Looking Back 5.12

Sums and differences of complex numbers are like the properties of exponents and radicals. To add or subtract with exponents, the bases must be alike. To add or subtract radicals, the radicand and indexes must be common. For complex numbers, the real parts are added to or subtracted from the real parts, and the imaginary parts are added to or subtracted from the imaginary parts.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Example 1: Simplify $(3 + 5i) + (2 + 7i)$.

Example 2: Simplify $(3 + 5i) - (2 + 7i)$.

This could also be done by aligning common terms using the vertical rather than horizontal method.

Example 3: Simplify the complex number. Combine any like terms.

a) $(3 - 7i) + (4 - 2i)$

b) $(7i - 6) - (-3 + 2i)$

Looking Ahead 5.12

To rationalize the denominator of a complex fraction, multiply by the complex conjugate.

Example 4:	Simplify $\frac{1}{3+i}$.

Example 5:	Find the reciprocal of $2i + 5$ and simplify it.

Example 6:	If $f(x) = \frac{1}{x}$, find $f(1 + i\sqrt{2})$.

Section 5.13 Products of Complex Numbers

Looking Back 5.13

Previously, in order to work with conjugates, you found the products of complex numbers. In this section, you will solve more complicated problems.

Example 1: Find the product.

$$5i(3i + \sqrt{5})$$

Example 2: Find the product. (You did this earlier in this module with radicals and conjugates.) Use the distributive property.

$$(3 + 6i)(4 - 2i)$$

Looking Ahead 5.13

Now you will use everything you have learned to find the sums of fractions.

Example 3:	$\frac{1}{2+3i}+\frac{4}{2-4i}$