# Algebra 2 Module 1 Foundations of Algebra

## Section 1.1 Graphs of Real Numbers

### Looking Back 1.1

Problem solving often involves algebraic reasoning. The foundation of algebraic reasoning involves thinking with real numbers. A good place to begin is with a review of the real-number system. Like our God, the Master behind the number system, our numbers are infinite and have no beginning and no ending.

**Real Numbers**

- Natural Numbers {1, 2, 3, 4, …}
- Whole Numbers {0, 1, 2, 3, 4, …}
- Integers {… -4, -3, -2, -1, 0, 1, 2, 3, 4, …}
- Rational Numbers $\left\{-\frac{1}{2}, \frac{2}{3}, 0, 1.5\right\}$

  These include all numbers that can be written as an integer divided by another integer that is not zero.
- Irrational Numbers $\left\{-\sqrt{2}, \sqrt{2}, \pi, e\right\}$

  These numbers cannot be written as one integer divided by another integer. The decimal representations of these numbers go on forever. They do not repeat or terminate.

### Looking Ahead 1.1

Each real number can be graphed on the number line because each point represents exactly one number.

The decimal 1.414213562373… is an approximation for $\sqrt{2}$. The decimal approximation can be used to locate the $\sqrt{2}$ on the number line, but $\sqrt{2}$ is used to mark the exact number.

If you want to find the distance between two numbers on the number line, simply take the absolute value of the difference of the two numbers.

For example, the distance between -1.8 and 2.3 is $|-1.8 - 2.3| = |-4.1|$ so the distance is 4.1.

| Example 1: | Locate 1.65 on the number line. Use absolute value to find a more precise approximation. |
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| Example 2: | Find the midpoint of 1.5 and 2.0 on the number line and use absolute value to find a more precise location for 1.65. |

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| Example 3: | List the following numbers in order from least to greatest: -5.1 $\sqrt{5}$ ¼ -22 |

| | |
|---|---|
| Example 4: | List the following numbers in order from greatest to least: ¾ $\sqrt{3}$ 0.35 -0.5 |

## Section 1.2 Properties of Real Numbers
## Looking Back 1.2

Knowing the properties of real numbers also helps in solving problems involving real numbers. The commutative property is closed under addition, but it is not closed under subtraction. It states that $a + b = b + a$.

Algebraic properties govern the use of the number system. These are like the scientific principles that God put in place to govern our universe. Things work properly, orderly and without chaos when these governing properties and principles are followed.

$$3 + 2 = 2 + 3$$
$$5 = 5$$

However, this does not work for subtraction.

$$3 - 2 \neq 2 - 3$$
$$1 \neq -1$$

The subtraction problem 3 – 2 could be changed to an addition problem written 3 + (-2) and the commutative property could be used.

$$3 + (-2) = (-2) + 3$$
$$1 = 1$$

Another helpful review is that of the properties of real numbers.

The following properties are true for every real number *a*.

| | |
|---|---|
| Reflexive Property | $a = a$ |
| Identity Property for Addition | $a + 0 = a$ |
| Inverse Property for Addition | $a + (-a) = 0$ |
| (Also called the Additive Inverse Property) | |
| Identity Property for Multiplication | $a \cdot 1 = a$ |
| Multiplication Property of Zero | $a \cdot 0 = 0$ |
| Multiplication Property of $-1$ | $a \cdot (-1) = -a$ (read "the opposite of $a$") |
| Inverse Property of Multiplication (Also called the Multiplicative Inverse Property) | $a \cdot \frac{1}{a} = 1$ |
| Transitive Property of Equality | If $a = b$ and $b = c$, then $a = c$ |
| Commutative Property for Addition | $a + b = b + a$ |
| Associative Property for Addition | $a + (b + c) = (a + b) + c$ |
| Commutative Property for Multiplication | $a \cdot b = b \cdot a$ |
| Associative Property for Multiplication | $a(bc) = (ab)c$ |

Distributive Property

$a(b + c) = ab + ac$ and $a(b - c) = ab - ac$

$ab + ac = a(b + c)$ and $ab - ac = a(b - c)$

Definition of Absolute Value

$$|a| = \begin{cases} a \text{ if } a > 0 \\ 0 \text{ if } a = 0 \\ -a \text{ if } a < 0 \end{cases}$$

Properties of Absolute Value

$|a| \geq 0$

$|-a| = |a|$

$|ab| = |a||b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Special Properties Concerning Zero

If $A \cdot B = 0$, then either $A =$ ____ or $B =$ _____.

The fractions $\frac{A}{0}$ and $\frac{B}{0}$ are undefined. It does _____ ___________.

The fraction $\frac{A}{0}$ is _____________, but the calculator types "UNDEF" for the ___________.

The fraction $\frac{0}{A}$ or $\frac{0}{B} =$ ________.

The fraction $\frac{0}{0}$ is indeterminate. It does not lead to a definite _______ or ___________.

Looking Ahead 1.2

Example 1: Use the properties of real numbers to simplify $(2 - 7 + 8)\frac{1}{3} + (8 + (-8))$.

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| Example 2: | Use mental math and the properties of real numbers to simplify $(2^3 + 2^3)\left(\frac{1}{2^3}\right) + \left(-\frac{5}{3} + 0.24\right) \cdot 0$. |

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| Example 3: | Evaluate $\frac{3a}{32-6a}$ when $a = 5$. |

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| Example 4: | Evaluate $(a^2b^2 - a^2b)\,\frac{1}{2b}$ when $a = 3$ and $b = 2$. |

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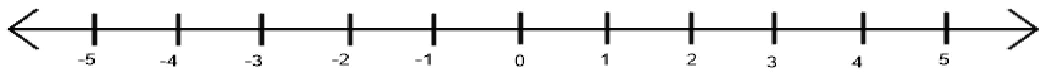
| Example 5: | The frame of a picture is 2 inches longer than it is wide. Find the area of the picture if the width is $12\frac{1}{2}$ inches. |

## Section 1.3 Adding and Subtracting Real Numbers

### Looking Back 1.3

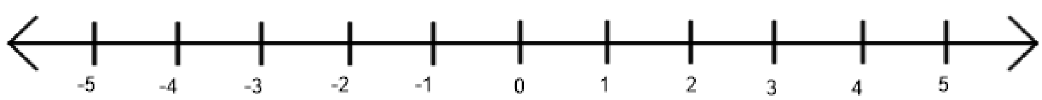
Just as there are properties of real numbers, there are rules for operating with real numbers. These same properties and rules apply to working with variables, since variables represent real numbers.

Addition and subtraction are inverses of one another. One operation undoes the other: 4 + 2 = 6 and 6 – 2 = 4. Adding a negative number is the same as subtracting a positive number: $a + (-\ b) = a - b$. All one really needs to know is the rules for addition of real numbers. Here is where absolute value can help. The difference between two numbers is the distance between the two numbers.




**Rules for Adding and Subtracting Real Numbers**

1. If $a$ and $b$ are positive, the sum is positive. $a + b = |a + b|$
2. If $a$ and $b$ are negative, the sum is negative. $a + b = -\ |a + b|$
3. If $a$ is positive and $b$ is negative, and the value of $a$ is greater than the value of $b$, then $a + b$ is a positive number. $a + b = |a| - |b|$
4. If $a$ is positive and $b$ is negative, and the value of $b$ is greater than the value of $a$, then $a + b$ is a negative number. $a + b = -\ (|b| - |a|)$




### Looking Ahead 1.3

Example 1: Simplify 18 + 22.3

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| Example 2: Simplify $-\frac{3}{4} + -\frac{1}{4}$. |

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| Example 3: Simplify $|5| - |2|$ |

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| Example 4: Simplify 5 + (-12) |

The first four examples were addition problems. In subtraction, a difference is to be found. Change subtraction to addition. Change the sign of the number following the operation symbol to its opposite and use the rules for addition of real numbers to find the sum.

| |
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| Example 5: Simplify 16 – (- 4) |

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| Example 6: Simplify -21 – 7 |

## Section 1.4 Multiplying and Dividing Real Numbers

### Looking Back 1.4

Based on the previous section, the definition for subtraction of real numbers is $a - b = a + -b$. The commutative and associative property are closed under addition, but not subtraction. The commutative property only holds true for subtraction if it is changed to addition.

Multiplication is distributive with both respect to addition and subtraction.

$a\,(b + c) = ab + ac$ and $a\,(b - c) = ab - ac$

$ab + ac = a\,(b + c)$ $ab - ac = a\,(b - c)$

Multiplying and dividing are inverses of each other. One operation undoes the other: $3 \times 4 = 12$ and $12 \div 4 = 3$.
Any division problem can be written as a multiplication problem: $9 \div 3 = 3$ and $9 \times \frac{1}{3} = 3$.

The rules for the multiplication of real numbers also hold true for the division of real numbers.

Rules for Multiplying and Dividing Real Numbers

1. If *a* and *b* are both positive, or if *a* and *b* are both negative, the product (also known as the quotient) is positive.

   $a \times b = |\,a \times b\,|$ and $\frac{a}{b} = \frac{|a|}{|b|}$

2. If *a* is positive and *b* is negative, then the product (quotient) is negative. The same holds true if *a* is negative and *b* is positive.

   $a \times b = -\,|\,a \times b\,|$ and $\frac{a}{b} = -\,\frac{|a|}{|b|}$

To simplify the multiplication and division of real numbers, remember that if the two signs are the same the product (quotient) is positive, but if the two signs are different the product (quotient) is negative.

Any even number of terms of a negative number has a product (quotient) that is positive, but any odd number of terms of a negative number has a product (quotient) that is negative.

### Looking Ahead 1.4

Example 1: Use real numbers to show that $-ab = (-a)\,b = a\,(-b)$. Let $a = 3$ and $b = 4$.

$-\,ab \rightarrow$

$(-a)\,b \rightarrow$

$a\,(-b) \rightarrow$

Example 3: Simplify $5\left(-\frac{1}{2}x + 4\right)$

Example 4: Use the properties of real numbers to show that $-(a + b) = (-a) + (-b)$

Example 5: Simplify $\left(-\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{4}t\right)$

Example 6: Evaluate $(2t)(t - 9)$ when $t = 4$.

Example 7: Simplify $(-2^3)(5 - 2(3))^2$

## Section 1.5 Using Symbols to Think Algebraically
### Looking Back 1.5

A previous problem in this module involved finding the perimeter of a picture and frame. The width was represented by $w$, and the length was represented by $l$. It is difficult to solve for two unknowns at one time. It is easiest to write one variable in terms of the other, then substitute a value for one variable to find the solution of the other.

If the length of the picture is 2" longer than the width, and the width is 14", then the length is 2" more, so the total length is 16." If the length of the picture is 19", and the width of the picture is 2" less, the total width is 17." If the width of the picture is $w$, the length of it is $w + 2$. If the length is $l$, the width is $l - 2$.

| Width ($w$) | Length ($l$) |
|---|---|
| 14 | 16 |
| 15 | 17 |
| 16 | 18 |
| 17 | 19 |
| $w$ | $w + 2$ |

| Length ($l$) | Width ($w$) |
|---|---|
| 19 | 17 |
| 17 | 15 |
| 15 | 13 |
| 13 | 11 |
| $l$ | $l - 2$ |

The width in terms of the length is $w = l - 2$. Values can be substituted for the length ($l$) to solve for the width ($w$). The length in terms of the width is $l = w + 2$. Values can be substituted for the width ($w$) to solve for the length ($l$).

The general formula for Area of a Rectangle is $A = l \cdot w$. Both length and width are needed to solve for area. However, if the width is known, but not the length, length in terms of width can be substituted into the formula $A = (w + 2) \cdot w$. Now the area can be found using only the known width.

The general formula for Perimeter of a rectangle is $P = 2l + 2w$. Both length and width are needed to solve for perimeter. However, if the length is known, but not the width, width in terms of length can be substituted into the formula and $P = 2l + 2(l - 2)$. This can be simplified to $P = 2l + 2l - 4 \rightarrow P = 4l - 4$. Now the perimeter can be found using only the known length.

Before going further, the rules for exponents will be reviewed.

Rules for Exponents

To add or subtract terms with exponents, the terms must have like bases and exponents. The base and exponent remain the same and the coefficients are added or subtracted.

$$a^x + a^x = 2a^x$$
$$a^x - a^x = 0$$

To multiply terms with exponents, the terms must have like bases, but the exponents may be different. The base remains the same, the exponents are added, and the coefficients are multiplied.

$$a^x \cdot a^y = a^{x+y}$$

To divide terms with exponents, the terms must have like bases, but the exponents may be different. The base remains the same, the exponents are subtracted, and the coefficients are divided.

$$\frac{a^x}{a^y} = a^{x-y}$$

To multiply a power to a power, multiply the exponents of each base by the exponent outside of parenthesis.

$$(ab)^x = a^x b^x$$

Examples of adding and subtracting with exponents:

$2a^4 + 2a^5$ These cannot be added. The bases are the same, but the exponents are different.

$-3m^2 - 8n^2$ These cannot be subtracted. The exponents are the same, but the bases are different.

$3a^2 + 5a^2 = (3 + 5)\, a^2 = 8a^2$ These can be added using the distributive property.

$-4m - 2m = (-4 - 2)\, m = -6m$ These can be subtracted using the distributive property.

Examples of multiplying with exponents:

$2^6 \cdot 2^2 = 2^{6+2} = 2^8$ The bases stay the same, and the exponents are added.

$m^4 \cdot m^3 = m^{4+3} = m$ The bases stay the same, and the exponents are added.

$3n^2 \cdot 2n^3 = 3 \cdot 2n^{2+3} = 6n^5$ The bases stay the same, the exponents are added, and the coefficients are multiplied.

Examples of dividing with exponents:

$\frac{2^6}{2^7} = 2^{6-7} = 2^{-1} = \frac{1}{2}$ The bases stay the same, and the exponents are subtracted.

$\frac{m^4}{n^4}$ Nothing can be simplified. The bases are different.

$\frac{2m^5}{-6m^3} = -\frac{m^{5-3}}{3} = -\frac{m^2}{3}$ The bases stay the same, the exponents are subtracted and the coefficients are divided.

Examples of raising a power to a power:

$(2 \cdot 3)^2 = (6)^2 = 36$ or $(2 \cdot 3)^2 = 2^2 \cdot 3^2 = 4 \cdot 9 = 36$

$(m \cdot n)^2 = (mn)(mn) = m^{1+1}n^{1+1} = m^2n^2$ or $(m^1n^1)^2 = m^{1\times2}n^{1\times2} = m^2n^2$

Looking ahead 1.5

Example 1: Write the expression for each underlined phrase in terms of width. Let $w$ = width.

a) The height is 2.5 cm. longer than the width.

b) The area is two and a half times the width.

c) The length is 2.1 cm. shorter than double the width.

d) The length is one-third of the width.

Example 2: Find the area of a triangle with a height that is 2 cm. longer than the base. Use the formula A = $\frac{1}{2}bh$.

If $b$ is the unknown base, find height in terms of the base to substitute for $h$ (use $h = b + 2$).

Or, let the height be $h$ and find the base in terms of the height to substitute for $b$ (use $b = h - 2$).

Example 3: Using the derived formulas for the area of the triangle in Example 2, find the area of the triangle if the base is 8 cm. Check your answer.

Example 4: Simplify $(2n)^4(3n)^2$

Example 5: Simplify $-\frac{1}{2}(2m^4)(n^3 + n)$ first then simplify $-\frac{1}{2}(2m^4)(n^3 + m)$ next.

Example 6: Simplify $\frac{2n^2 - 3m^4}{6mn}$ first then simplify $\frac{2n^2(-3m^4)}{6mn}$ next.

## Section 1.6 Solving Equations with One Variable

### Looking Back 1.6

Many of the mathematics symbols used in algebra were developed fairly recently, being within the last 400 to 600 years. The plus sign (+) was believed to be used first by Nicole d'Oresme as an abbreviation for *et*, which means "and" in Latin. It was used in the book *Algorismus Proportium* during the fourteenth century (the 1300s). The plus (+) and minus (-) symbols were both used by Johannes Widman in the late 1400s to refer to an excess or deficit in the weight of boxes at his business.

William Aughtred used the "×" symbol for multiplication in his book *Clavis Mathematical* (*Key to Mathematics*) in 1628 in London. It was called Saint Andrews Cross. Gottfried Wilhelm Leibniz used a dot (·) for multiplication. He did not like the symbol "×" because he said it looked too much like the letter "x." The division symbol (÷) is called the *Obelus* and was first used in the 1600s to represent fractions.

All these symbols are still being used in algebra today along with the algebraic properties to solve equations with unknown variables. These symbols are used to write rules that guide us through the steps of an equation.

You have heard me mention The Golden Rule of Algebra. It is derived from what is called the Golden Rule of God. God tells us to love the Lord our God with all our hearts, souls and minds. He also tells us to love our neighbors as ourselves and to treat others as we would like to be treated. Following this keeps life centerd and balanced.

The Golden Rule of Algebra means that whatever we do to one side of an equation we must do to the other side of an equation to keep it balanced. This keeps the right side of an equation equal to the left side.

In the equation $2x + 3 = 13$, the operations of multiplication and addition are used. The equation asks what number when doubled and added to 3 yields 13. We can use reverse thinking to "undo" the operations and solve for $x$. Before $x$ was doubled the value was half of that, and before 3 was added it was three less.

When simplifying, we first combine things by grouping them and simplifying within the grouping symbols. When working in reverse we are going to "ungroup" them.

- Undo any operations that are outside the grouping symbols.
- Undo any exponents with their inverses (the inverse of a square is a square root, the inverse of a cube is a cube root, and the inverse of an *n*th power is an *n*th root).
- Undo anything left in parenthesis. Going backwards is working outside to inside.

### Looking Ahead 1.6

To solve for an unknown variable means to find out what it equals. It must be isolated or left on a side of the equation alone with everything else on the other side of the equal sign.

Example 1: Solve for $m$ in the equation $-22m + 11 = 55$. Check your answer.

Example 2: Solve for $m$ in the equation $-6 + \frac{m}{13} = -3$. Check your answer.

Example 2: Solve for $m$ in the equation $-6 + \frac{m}{13} = -3$. Check your answer.

Example 3: Solve for $t$ in the equation $(t - 8)^2 = 16$. Check your answer.

Example 4: Solve for $p$ in the equation $11 = -2p - 5$ Check your answer.

Example 5: Solve for $s$ in the equation $(s + 5)^3 + 4 = 31$. Check your answer.

Example 6: Solve for $d$ in the equation $\frac{1}{2}(3 + d)^2 - \frac{3}{2} = -1$. Check your answer.

## Section 1.7 Using Equations to Solve Problems
## Looking Back 1.7

Algebra can be used to solve many real-world problems. In *Pre-Algebra,* we solved a problem of antiquity and found Diaphantes' age using algebra. In *Algebra 1*, we solved problems involving speed of light and speed of sound to find the pitches of tuning forks. We also solved problems involving the projectile motion of baseballs and rockets modeled by quadratic equations. The geometry problems involving perimeter and area in the previous section were solved by using algebraic techniques.

Setting up a problem correctly is a very important step to solving a problem. Since addition is commutative, the order is not important. The sum of a number and the cube of that number may be written as follows:

$$n + n^3 \qquad \text{or} \qquad n^3 + n$$

However, the difference of two numbers is not commutative. The difference of a number and that number cubed is written as follows:

$$n - n^3$$

If the algebraic expression is $n^3 - n$ then the written expression is the following: "the difference of a number cubed and that number." Order is important in subtraction.

If a number and double that number is $-18$, it may be written as $n + 2n = -18$ or $2n + n = -18$.

$$n + 2n = -18 \quad \text{or} \quad 2n + n = -18$$
$$n(1 + 2) = -18 \quad \text{or} \quad n(2 + 1) = -18$$

$$n \cdot 3 = -18$$
$$3n = -18$$
$$\frac{3n}{3} = -\frac{18}{3}$$
$$n = -6$$

The answer can be checked by substituting -6 in for *n* in the original equation.

$$n + 2n = -18 \quad \text{or} \quad 2n + n = -18$$
$$-6(1 + 2) = -18 \quad \text{or} \quad -6(2 + 1) = -18$$

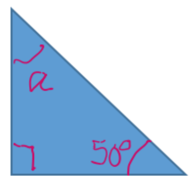
$$-6 \cdot 3 = -18$$

Below are multiple ways to the write the equation.

$$-6 + 2(-6) = -18 \quad \text{or} \quad 2(-6) + -6 = -18$$
$$-6 + (-12) = -18 \quad \text{or} \quad (-12) + -6 = -18$$

Looking Ahead 1.7

Example 1: Find the degree of the unknown acute angle *a*. The sum of the angles of a triangle equals 180.°




Example 2: There are 36 delegates attending a conference. The ratio of men to women must be 3:1 to accommodate housing. How many men and how many women may attend the conference?

Example 3: Jean and Bessie live 25 miles apart. Jean leaves her house on her moped at 11:00 a.m. and travels east to visit Bessie at 80 mph. Bessie rides her bike west so she can meet Jean for lunch. She bikes at a rate of 5 mph. She leaves her house at the same time as Jean. What time will they meet for lunch? Draw a diagram to help.

| Rate | Time | Distance |
|---|---|---|
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| Example 4: | Regan and Emma competed in a science competition. They scored 4 points less than the team presenting to the right of them and 3 points more than the team to the left of them. All three teams scored 136 points combined. How many points did each team score? |

| | |
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| Example 5: | The pep club sold student rally tickets for $3.00 each and thirty-three fewer adult tickets for $6.00 each. They made $540.00 for the rally. How many student tickets did they sell? How many adult tickets did they sell? |

Section 1.8 Solving Inequalities in One Variable
Looking Back 1.8

An equation uses an equal sign. An inequality uses one of the following four signs:

$\geq$ Greater Than or Equal To

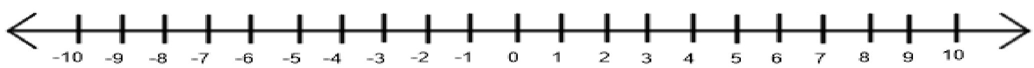
$\leq$ Less Than or Equal To

> Greater Than

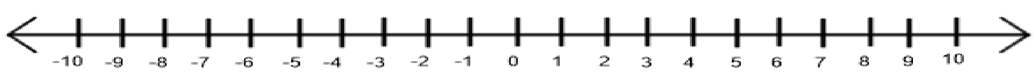
< Less Than

If $n > 3$, then $n$ is any number greater than three. All the listed numbers, 3.1, $4\frac{1}{2}$, 5.293, and $\sqrt{37}$ are greater than three, but not equal to or less than three. This can be written as "$n \nleq 3$."

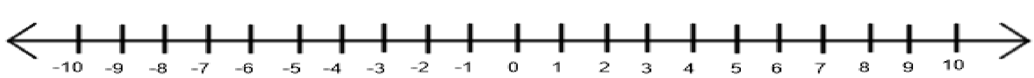
On the number line, 3 is left open with a circle to show that it is the start value, but is not included. All numbers to the right of 3 are shaded to show inclusion.



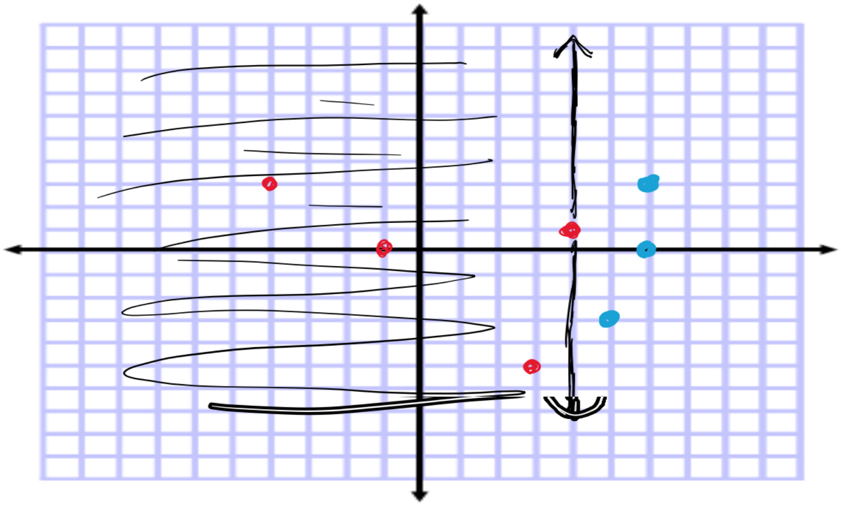

If $m \leq -2.1$, then $m$ includes - 2.1 and every number less than - 2.1. The circle at - 2.1 is darkened or shaded in to show that - 2.1 is a solution of the inequality. Other solutions are any numbers shaded to the left of - 2.1, such as $-8$, $-2\frac{1}{2}$, and - 179.




If $x < 4$, it is evey number to the left of 4. If $x \geq 7$, it includes 7 and every number to the right of 7.



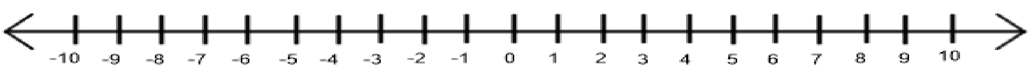

The inequality $x \leq 4$ can be graphed on the number line in one-dimension or it can be graphed on the coordinate plane in two-dimensions. All values to the left of $x = 4$ are shaded in to show inclusion and the line $x = 4$ is darkened to show that any point on that line is also in the solution set of the inequality. The coordinates in red have $x$-values that are equal to or less than 4: (4, 1), (3, - 5), (- 1, 0) and (-4, 3). The coordinates in blue have $x$-values greater than 4 and are in the non-included region of the coordinate plane: (5, - 3), (6, 0), and (6, 3).



Looking Ahead 1.8

Sometimes inequalities with only one variable must be simplified or solved so that the variable is isolated on one side and all other numerical values are combined on the other side. The same steps used to solve equations are used to solve inequalities.

Example 1: Graph the inequality $\frac{s}{1.4} > 1.2$ on the number line.



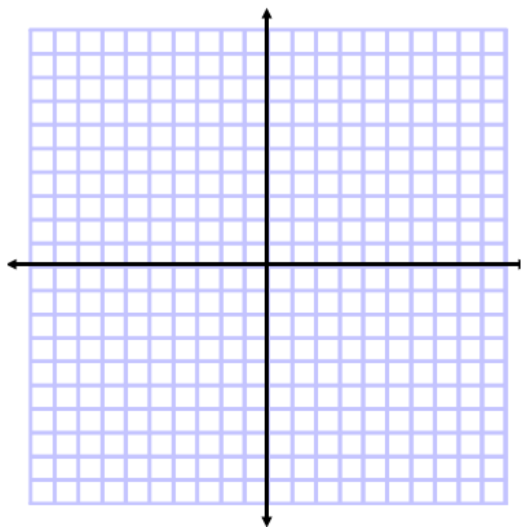

Example 2: Demonstrate that the solution to the inequality in Example 1 is correct. Let $s$ = - 1, 1.68 and 3.

We cannot test every value for the solution set because there are infinite numbers that work. However, only one counterexample is needed to demonstrate a value that is not in the solution set.

God is spirit. Like the number system, God is infinite. To say that God does not exist, one would have to look under every rock and plant to prove it. There are no counterexamples to God's existence. One can observe from trees and flowers that God does exist. It is impossible to prove that He does not exist, but many would agree that there is much evidence that He does exist. Not only does God exist but He is infinite just like inequalities.

Inequalities are solved just like equalities only there are infinite solutions instead of only one. To solve for the unknown variable, isolate it on one side of the inequality sign.

Example 3: Solve the inequality $2x + 3 \leq -5$ and graph the solution set on the coordinate plane.



Just as there are properties of equalities, there are properties of inequalities.

Properties of Inequalities

For all real numbers $a$, $b$, and $c$:

| | | | |
|---|---|---|---|
| Either $a < b$ , | $a = b$ | or | $a > b$ |
| $b < c$, | $b = c$ | or | $b > c$ |
| $a < c$, | $a = c$ | or | $a > c$ |

Addition and Subtraction Property

| | | |
|---|---|---|
| $a + c < b + c$ | if | $a < b$ |
| $a + c = b + c$ | if | $a = b$ |
| $a + c > b + c$ | if | $a > b$ |

Multiplication and Division Property

If $a < b$ and $c > 0$, then $ac < bc$
If $a < b$ and $c < 0$, then $ac > bc$

Dividing by a whole number is the same as multiplying by a fraction.

$2 \div -2 = -1$ or $2 \cdot \left(-\frac{1}{2}\right) = -\frac{2}{2} = -1$

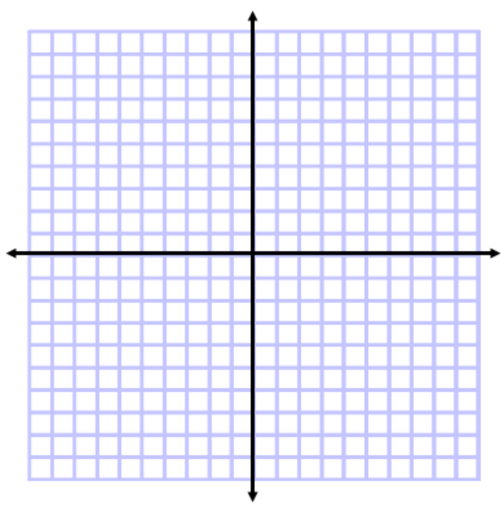
Multiplying or dividing by a negative number changes the direction of the inequality sign.

$$-\frac{x}{3} > 4$$

$$-3\left(-\frac{x}{3}\right) < 4(-3)$$

$$x < -12$$

Example 4: Solve the inequality - 5(*x* – 3) + 4 > 3(- 2) and graph the solution on the coordinate plane.



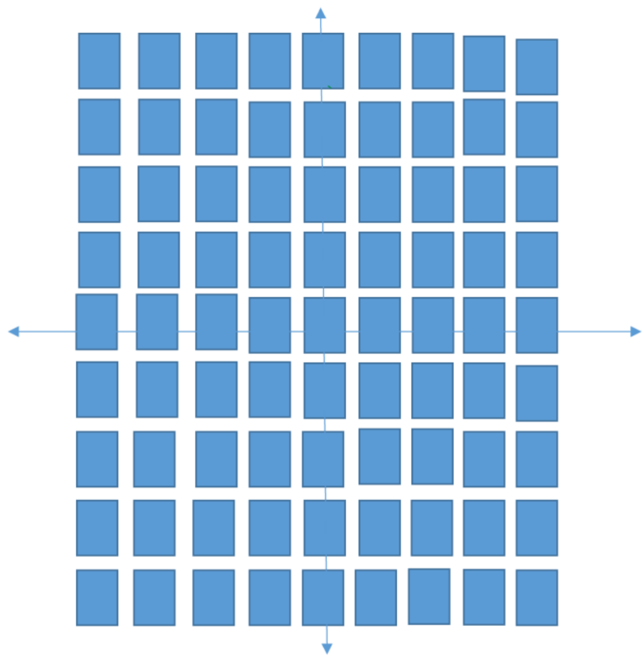
## Section 1.9 Solving Inequalities in Two Variables

### Looking Back 1.9

Inequalities in one variable are graphed on the number line, the one-dimensional plane. Inequalities in two variables are graphed on the coordinate plane, which represents two dimensions.

Let the boxes on the coordinate grid below represent all the ordered pairs of integers in the domain $-4 \leq x \leq 4$ and the range $-4 \leq y \leq 4$. Is (3,4) a solution of the equation $y = 2x + 1$?

The ordered pair (3,4) is not a solution to the equation $y = 2x + 1$. However $4 < 2(3) + 1$. A red less than symbol (<) will be inserted in the box at the ordered pair (3,4).



Is the ordered pair (- 2, 0) a solution of the equation $y = 2x + 1$?

The ordered pair (- 2, 0) is not a solution to the equation $y = 2x + 1$. However $0 > 2(-2) + 1$. A green greater than symbol (>) will be inserted in the box at the ordered pair (-2,0).

Let's try one more. Substitute (0, 1) in for the values $x$ and $y$ and insert the correct equality or inequality symbol in the circle to make the statement true: $y$ ◯ $2x + 1$. Then put that symbol in the coordinate grid at the point (0 , 1).

Stop the video at this point and complete the boxes in the coordinate plane. Insert the correct symbol that makes a correct statement. Use a red less than symbol (<), a black equal to sign (=), or a green greater than symbol (>) to make the statement $y$ ◯ $2x + 1$ correct.

When you have completed every box on the coordinate grid start the video over again to check your solution.

There is a pattern here. The equal signs form a diagonal line. This is all the points where $y$ is equal to $2x + 1$.

Below and to the right of the equal signs, or boundary line $y = 2x + 1$, are less than signs (<). Normally, we shade below the line to demonstrate this. These are all the points where $y < 2x + 1$.

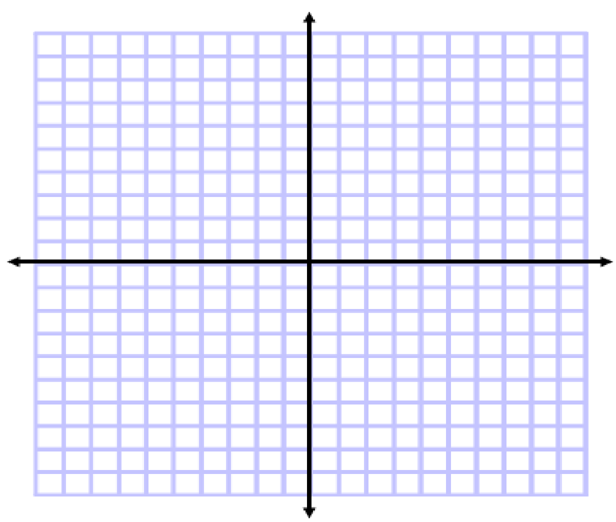
Above and to the left of the equal signs, or boundary line $y = 2x + 1$, are greater than signs (>). Normally, we shade above the line to demonstrate this. These are all the points where $y > 2x + 1$.

<u>Looking Ahead 1.9</u>

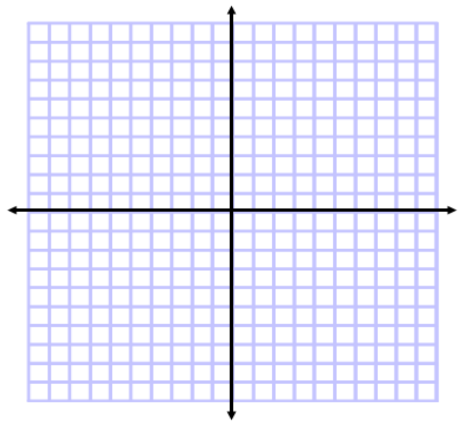
To graph an inequality on the coordinate plane, solve for $y$ in terms of $x$. This means to convert the equation from general or standard form to slope-intercept or $y$-intercept form. Draw the boundary line just as you would for an equation by identifying the $y$-intercept and slope. Draw a point where the inequality boundary line crosses the $y$-axis. From there, move vertically (up or down) for the rise and horizontally (right or left) for the run and draw a second point. Connect the two points and you have the boundary line.

If the inequality symbol is greater than or equal to or less than or equal to ($\geq$ or $\leq$), the boundary line is included and is shaded in to make a solid line. This means the line and everything above or below the line that is shaded is included in the solution. If the inequality symbol is greater than or less than ($>$ or $<$), the boundary line is not included and is dotted to make a dashed line. Only everything above or below the line that is shaded is included in the solution.

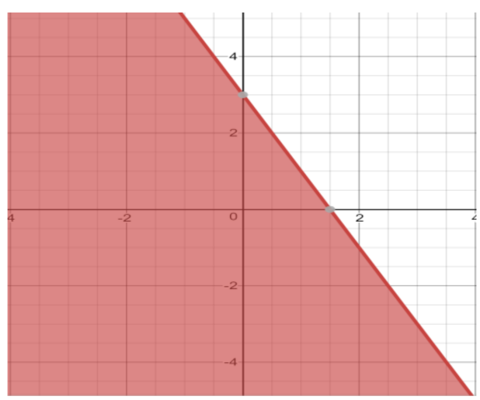
Example 1: Graph the inequality $22x + 11y \geq 33$.



Example 2: Graph the inequality $y < -\frac{1}{2}x + 4$.



Example 3: Write the inequality for the graph given.

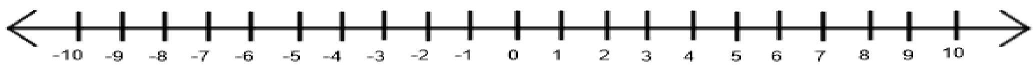



Example 4: A medium pizza costs $6.25. That comes with cheese only. Each additional topping costs $0.50. Imagine you and your friends have $10.75 together. Write an inequality to find the maximum amount of toppings you can order for your pizza.

Section 1.10 Solving Conjunctions, Disjunctions, and Absolute Value Inequalities
Looking Back 1.10

When two sentences are combined using the word "and" it is called a conjunction. When two mathematical inequalities are combined with the word "and" both must be true at the same time. The solution set is the combination of the inequalities.

The solution set of $p > -1$ and $p < 2$ is all the numbers between -1 and 2, not including -1 and 2.

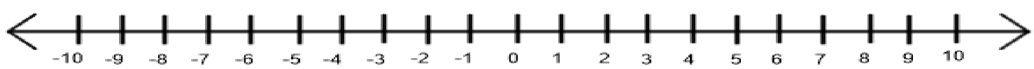



The intersection of these two sets is the conjunction that shows the solution, both one and the other at the same time.

There are several solutions: $p = 0$ , 1.2, or $-\frac{1}{3}$. They are all in the intersection of the two inequalities and can be combined and written as $-1 < p < 2$. The inequality $-1 < p$ is the same as $p > -1$. When the order of the inequality is changed, the inequality sign must be changed to its' opposite. For example, $p \leq 3$ may be written as $3 \geq p$.

When two sentences are combined using the word "or" it is called a disjunction. When two mathematical inequalities are combined with the word "or" only one of them can be true, either one or the other.

The solution set of $p < -1$ or $p > 2$ is all the numbers less than - 1 or greater than 2.




The values that work for $p$ are - 1.9, $-2\frac{1}{5}$, 4 or 99 to name a few. All of these numbers are in the shaded region. That is both of the two inequalities and can only be written as above or as $-1 > p$ or $2 < p$. Though the latter form is not very helpful in understanding and graphing the problem.

To solve and graph the inequality $4 \leq 2z + 5 < 10$, split it into two inequalities using the word "and."

Solve each inequality separately. Combine these into one inequality with the variable isolated in the middle.

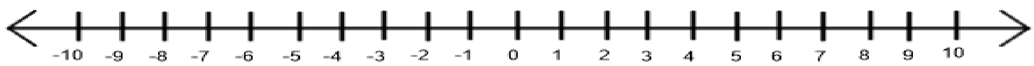
$4 \leq 2z + 5$ and $2z + 5 < 10$


This is the intersection of $z \geq -\frac{1}{2}$ and $z < \frac{5}{2}$.

Note the circle on the left is closed to show inclusion because of the ≤ sign. The circle on the right is open to show non-inclusion because of the > sign. It can be combined and written $-\frac{1}{2} \leq z < \frac{5}{2}$. It can then be graphed as the intersection of the two inequalities.

The disjunction $4 - 3n \leq 1$ or $5n - 1 < 3 + n$ is already split and is solved as two inequalities.

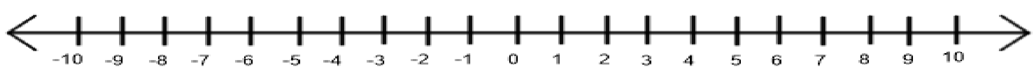
$4 - 3n \leq 1$ or $5n - 1 < 3 + n$




The solution set ends up being all real numbers since the entire number line is included.

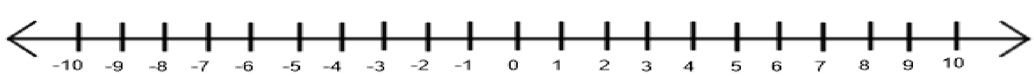
<u>Looking Ahead 1.10</u>

Absolute value inequalities are also solved using conjunctions or disjunctions. Absolute value means the distance from zero. When $| s | = 1$, then $s = 1$ or $s = -1$. Both are one unit from zero. When $| s | = 3$, then $s = 3$ or $s = -3$. Both are three units from zero.

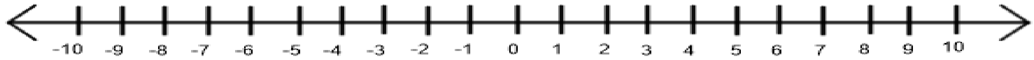



Therefore, $| s - 2 | = 5$ means that 2 subtracted from a number is five spaces from zero. We know that $7 - 2 = 5$ so 7 is one solution for $s$. The previous solutions included a positive and negative solution. What negative number is five spaces from zero when 2 is subtracted from that number? We know that both 5 and -5 are five spaces from zero so we can drop the absolute value sign, set up two equations and solve.

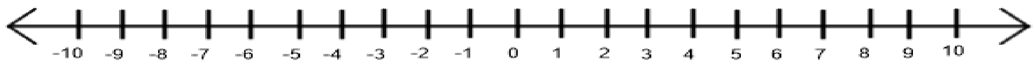
Example 1: Write $| s - 2 | = 5$ as a disjunction and solve. Check to see if the two solutions work.




What does $| s | > 1$ mean? This is a number such that the space between the number $s$ and zero is greater than 1. That occurs when $s > 1$ or $s < -1$.



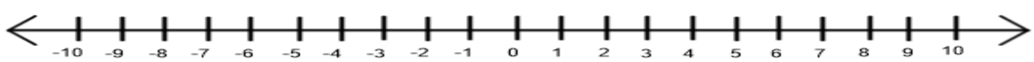

| Example 2: Using the information from $| s | > 1$, write the inequality $| s – 2 | > 1$ as a disjunction and solve. |
|---|




Check one included value on each side to verify the solutions. An exhaustive check is not possible since there are infinite solutions.

What does $| s | < 1$ mean? How can this be written as a conjunction? This means the distance between zero and the number must be less than 1.

| Example 3: Graph the solution to the inequality $| s | < 1$. Check the solutions. |
|---|

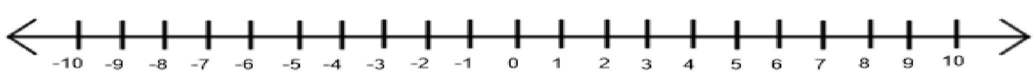



Therefore, the conjunction is $- 1 < s < 1$.

When the absolute value has a greater than or a greater than or equal to sign, set up a disjunction using the word “or” and solve. Drop the absolute value bars. Keep the original problem the same. Change the direction of the inequality symbol and the value after it to its opposite for the second equation following the disjunction.

When the absolute value has a less than or less than or equal to sign, set up a conjunction using the word “and” and solve. Drop the absolute value bars. Keep the original problem the same. Change the direction of the inequality symbol and the value after it to its opposite for the second equation following the conjuntion.

| Example 4: Graph the solution to the inequality $| 6 – 4n | \leq 10$. Check the solutions. |
|---|



## Section 1.11 Solving Systems of Equations Using Graphs

### Looking Back 1.11

Earlier in the module, you graphed an equation involving one variable on the coordinate plane. You also graphed an inequality involving two variables on the coordinate plane and used shading to show the solutions of the inequality.

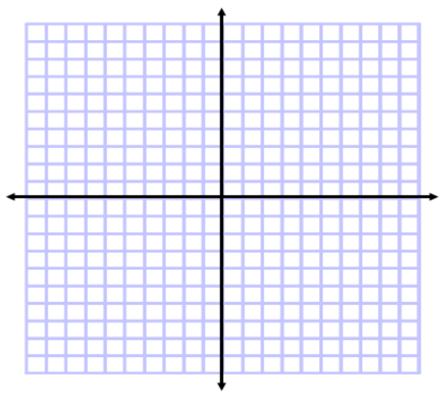
In this section, you will find solutions for more than one equation or more than one inequality. This is called solving a system of equations or a system of inequalities.

We will start by looking at solving two equations using graphing and end by solving two inequalities using graphing.

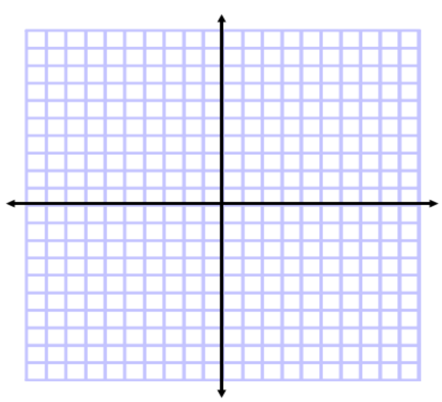
Since two lines can be graphed on the same coordinate plane, the intersection of those lines would be the point common to both lines. This point is the shared solution for both equations.

### Looking Ahead 1.11

Example 1: Find the intersection of $y = -3x + 7$ and $y = -2x + 3$ by graphing both equations on the coordinate plane and finding the intersection of the two lines.



Example 2: Find the intersection of the equations $y = -2 - 4(x - 5)$ and $8x + 2y = 12$ on the graph.



The two lines are parallel. They will never intersect in the two-dimensional plane. Therefore, they have no common solution. There is not one solution that satisfies both equations at the same time.

Example 3: Find the intersection of the two graphs of the equations $-2x + 3y = 4$ and $-4x + 6y = 8$. Firstly, convert both standard form equations to *y*-intercept form in order to graph.

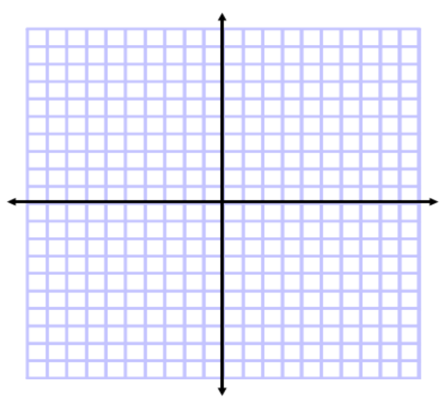
There is no need to graph the equations. These are the same line. They have the same slope and the same *y*-intercept. Therefore, they have infinite solutions. One line lies on top of the other.

Example 4: Graph the system of inequalities. Darken in the points of intersection. These vertices are called the corner points and often represent constraints, such as minumum or maximum profits.

$y \leq -\frac{1}{2}x + 2$

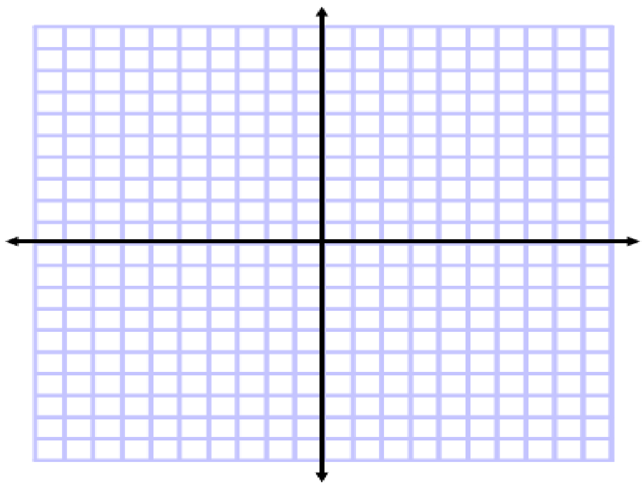
$y \geq \frac{2}{3}x - 1$

$x \geq -3$



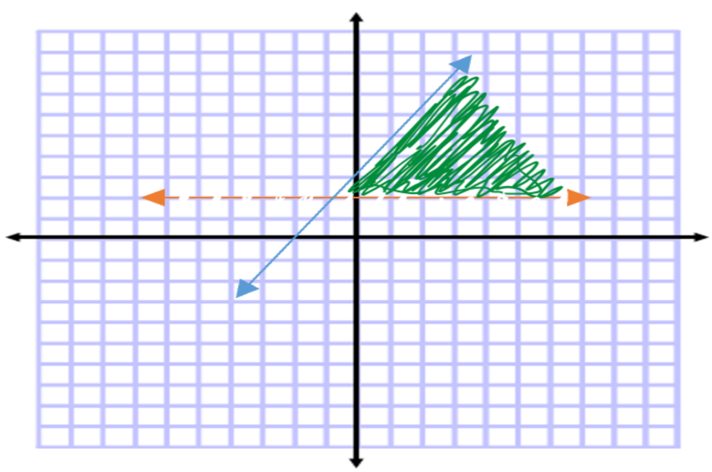
Example 5: Solve the problem by setting up two inequalities and find the overlapping area using graphing.

Jocelyn's bracelet company is selling bracelets and giving a $10.00 gift certificate as a prize. The profit for each beaded bracelet is between $1.50 and $2.00. Find the feasible region for the possible profit. The feasible region is the area that has solutions that work for both equations at once. Let *s* represent the number of bracelets sold *p* represent profit.



Is it possible to make a $20.00 profit if 15 bracelets are sold? The point representing this is (15, 20). Any (*x*, *y*) coordinate on the plane is (*s*, *p*) where *s* is the indendent variable for the number of bracelets sold and *p* is the dependent variable for the profit earned based on the number of bracelets sold. Try the point (15, 20) in both inequalties and see if it holds true for both.

Example 6: Write the system of inequalities that describe the shaded region.



## Section 1.12 Solving Systems Using Substitution
## Looking Back 1.12

When one value is substituted in for another, the values must be equal. If *a* = *b* and *b* = *c*, then we can substitute *c* for *b* and say that *a* = *c*. Since *a* and *c* are both equal to the same value, in this case *b*, they are equal to each other.

In Examle 1 of the last section, a graph was used to find the intersection of the equations $y = -3x + 7$ and $y = -2x + 3$. Both graphs are linear but have different slopes and different *y*-intercepts. Therefore, they intersect at one point. The point (4, -5) is the (*x*, *y*) point where the two lines intersect and the one solution to both equations.

That is the point where *x* is equal to *x* and *y* is equal to *y* on both lines. Since the equations are in *y*-intercept form, and equal to the same number at the point of intersection, namely *y*, both are equal to each other.

If $y = -3x + 7$
and $y = -2x + 3$
then $-3x + 7 = -2x + 3$
since $y = y$

The above equation has everything in terms of *x*. Since there is only one variable in the equation, *x*, we can find the solution for *x* using algebra.

$$-3x + 7 = -2x + 3$$

Now we can substitue the solution for *x* into either of the original two equations, since they are both equal to *y*, and find the value of *y* at the point of intersection.

$y = -3x + 7$ $y = -2x + 3$

Check the point in the equations to make sure it works for both.

$y = -3x + 7$ $y = -2x + 3$

One method for solving systems of equations is graphing. It can be difficult by hand, but much easier using a graphing utility. Substitution is another method for solving systems of equations.

## Looking Ahead 1.12

In Example 3 of the previous section, you used graphing to find the solution for the two equations $-2x + 3y = 4$ and $-4x + 6y = 8$.

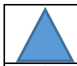
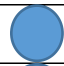
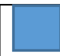
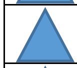
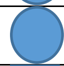
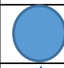
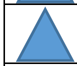
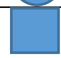
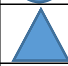
| | |
|---|---|
| Example 1: | Use the substitution method to find the solution to the equations $-2x + 3y = 4$ and $-4x + 6y = 8$. Convert both standard form equations to $y$-intercept form first. |

| | |
|---|---|
| Example 2: | The Bogus Business is buying new software for the employees. Company Laugh-a-Lot offers the software for $12,000.00 plus $600 for each additional site license. Company Jokester offers the software for $3,500.00 plus $1,200 for each additional site license. Write two equations to represent the cost of each software and determine which software costs less for 10 employee site licenses. |

Substitution can also be used to solve a system of three equations. This is a point where three lines meet in the three-dimensional $x$-$y$-$z$ plane. Let's try it first by using symbols to represent variables.

| | |
|---|---|
| Example 3: | If △ + △ = 8, find 3△ - △ |

Let’s use an array with a row sum and column sum to try and find the values of the missing shapes.

| | | | |
|---|---|---|---|
|  |  |  | 9 |
|  |  |  | 8 |
|  |  |  | 8 |
| 6 | 10 | 9 | |

First, we will find the value of the triangle.

Example 4: Use substitution to find the values of the circle and square in the above array.

Example 5: Use the substitution method to solve the system of three equations.

$x + y + z = 9$
$x + 2y$
$2x + \quad z = 8$

Step 1: Solve for $y$ in terms of $x$ in the second equation.

Step 2: Solve for $z$ in terms of $x$ in the third equation.

Step 3: Substitute the expressions for $y$ and $z$ in the first equation and solve for $x$.

Step 4: Substitute the numerical value for $x$ in the second equation and solve for $y$.

Step 5: Substitute the numerical value for $x$ in the third equation and solve for $z$.

Step 6: Check to see if the values for $x$, $y$, and $z$ work for all three equations.

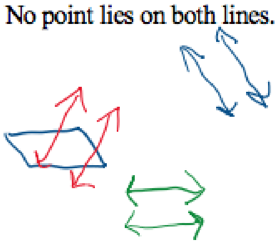
The last two examples both used substitution to solve systems. However, in Example 4, we used $x$ for the triangle, $y$ for the circle, and $z$ for the square. This also introduced the process of elimination, which we will use in the next and final section of the module to solve a system of equations (we will actually use substitution with elimination). In the Practice Problem section, we will further investigate the three-dimensional plane.

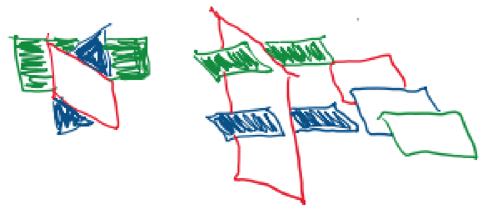
## Section 1.13 Solving Systems Using Elimination
## Looking Back 1.13

Solving systems of equations with three variables is very similar to solving systems of equations with two variables. There may be one solution, no solution or infinite solutions.

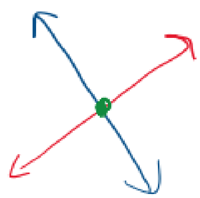
The graphs of these are different. Two lines intersect in a point, but two planes intersect in a line.

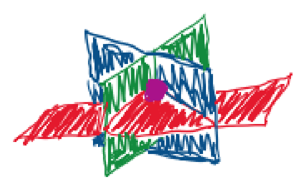
| Two Variables<br>Two Equations | | Three Variables<br>Three Equations |
|---|---|---|
| The lines are parallel. | | The planes are parallel. |
| No point lies on both lines. | ←No Solution→ | At most, two planes intersect one another. |



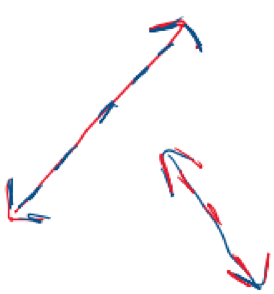


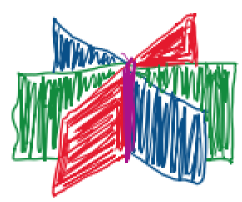
| | | |
|---|---|---|
| The lines intersect in one point common to both. | ←One Solution→ | The planes intersect in one point common to all three. |





| | | |
|---|---|---|
| The lines are collinear. | ←Infinite Solutions→ | The planes are coplanar. They intersect at all points on a common line. |





Looking Ahead 1.13

This time we will revisit three equations involving three variables and use the method of elimination.

We know that we can perform any operation of an equation and it remains equal if the operation is done to both sides of the equation. Two equations may be added or subtracted because you are adding equal things to both sides.

Example 1: Use elimination to find the values of two numbers if the sum of the numbers is 16 and the difference of the numbers is - 4.

Example 2: Use elimination to solve the system of equations.
$x + 2y = 10$
$2x + 2y = -4$

Example 3: Use the elimination method to find the values of $x$, $y$ and $z$. Find the ordered triple that is a solution to all three equations.
$x + y + z = 9$
$x + 2y = 8$
$2x + z = 8$

Step 1: Subtract the third equation from the first equation.

Step 2: Add the second equation to the equation derived in Step 1. That will eliminate $x$ and you can solve for $y$.

Step 3: Substitute the value of $y$ in the second equation and solve for $x$.

Step 4: Substitute $x$ in the third equation and solve for $z$ or substitute $x$ and $y$ into the first equation and solve for $z$.

Example 4: Camilla works at a jewelry shop. She has a $1,300.00 budget to stock her display cases with copper, tin, and bronze jewelry. She has enough room to display 64 pieces of jewelry. The most popular is bronze and the least popular is tin so she wants to stock triple the amount of bronze pieces as tin pieces. The cost of tin jewelry is $10.00 each piece. The copper jewelry is double the amount of tin. Bronze jewelry is $24.00 for each piece. How much of each piece can she display in her cases?

Let $t$ = tin, $c$ = copper, and $b$ = bronze.

Set up three equations and solve. Find out how many of each piece of jewelry Camilla needs to stock in her display cases so that she will meet all of the constraints.

Step 1:

Step 2:

Step 3:

Step 4:

Example 5: John is a fused glass artist and he makes 6 plain and 2 decorative ornaments in one hour. He must make at least 36 ornaments in a work shift of at most 8 hours. Write a system of inequalities for the time spent on each ornament. The artist makes $20.00 an hour on plain ornaments and $25.00 an hour on decorative ornaments. Write a function for the total profit made on the ornaments. How many hours must the artist work to maximize the profit. What would the profit be?