Module 6: Quadratic Functions

Section 6.1 Forms of the Quadratic Equations

Looking Back 6.1

Now that we understand squares and square roots, we can begin to investigate quadratic equations. A quadratic equation is an equation whose highest degree monomial term is a square. The standard form (sometimes called the *general form*) of a quadratic equation is $ax^2 + bx + c = y$ in which $a \neq 0$ and a, b, and c are all real numbers. It has a quadratic term, a linear term, and a constant term. The coefficient of b or c may be 0 so a quadratic equation could be a monomial, binomial, or trinomial.

$$y = ax^2 + bx + c$$

Quadratic Term Linear Term Constant

This is called the standard form because it is the way quadratic equations typically appear. Standard form quadratic equations can be factored into the product of two linear factors in which $a \neq 0$ and $b \neq 0$ $(ax + c)(bx + d) = y$. This form helps us see the x-intercepts (where the equation intersects the x-axis), so we can graph the equation.

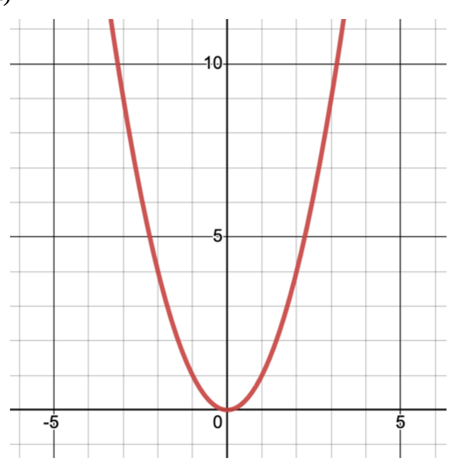
Standard form equations can also be factored into another form, called the vertex form (sometimes called the *graphing form*), which is $a(x - h)^2 + k = y$ in which $a \neq 0$ and a, h, and k are real numbers. This helps us see where the vertex (the maximum or the minimum point of the graph) lies and is another form used to graph the equation. We will study these forms later in this module.

There are many forms of quadratic equations and we will learn about all of them in this module. However, the most basic form of quadratic equations is called the parent function in which $a = 1$, $b = 0$, and $c = 0$. The equation $y = x^2$ is called the parent function of a quadratic equation because $a = 1$, $b = 0$, and $c = 0$.

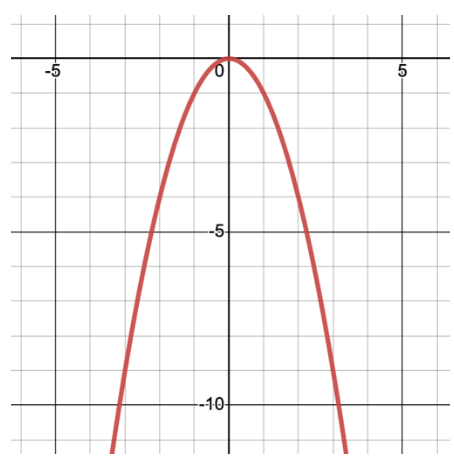
Looking Ahead 6.1

Example 1: Below are two quadratic graphs that we have previously investigated. The one on the left opens upward and the one on the right opens downward. Therefore, either of these graphs passes the origin. Moreover, both of these are parabolas. Write the equations for the graphs below.

a)



b)



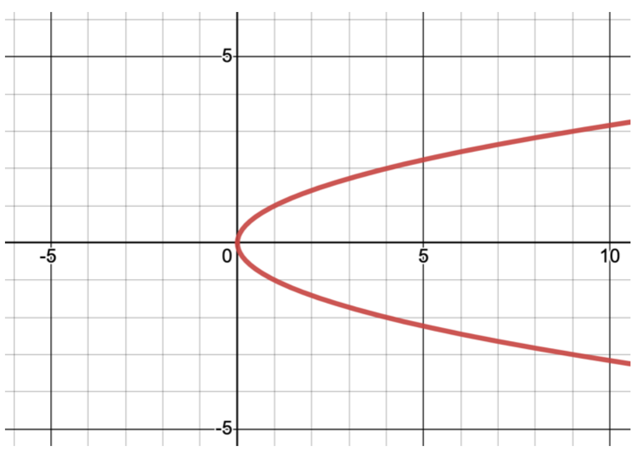
The above graphs individually are said to be symmetric in the y-axis. One side of the parabola is a reflection of the other side of the parabola over the y-axis; therefore, the line $x = 0$ is called the axis of symmetry (line of symmetry).

The graph on the left is $y = x^2$. It opens upward. What change do you think was made to the equation on the right that made the graph open downward?

Notice that $y = -x^2$ is in Quadrants III and IV so that when the input (x) is positive or negative, the output (y) is negative. What sign would change the input so the output becomes its opposite? The equation on the right is $y = -x^2$.

"Lazy" parabolas open to the left or right. They are not quadratic equations but square root equations. They are inverses of quadratic equations because a square root undoes a square. Square root equations are not defined as functions because they do not pass the Vertical-Line Test, and they do not pass the Vertical-Line Test because one value of x gets mapped onto more than one value of y. If you draw a vertical line through the graph, it increases through more than one point. Only at $(0, 0)$, the origin, does a vertical line pass through one point. These square root equations that are relations can be made into square root equations that are functions by putting restrictions on the range.

Example 2: Below on the left is a graph of the square root function. Write the domain and the range (the possible input and output values) of the graph and tell if it is a function.



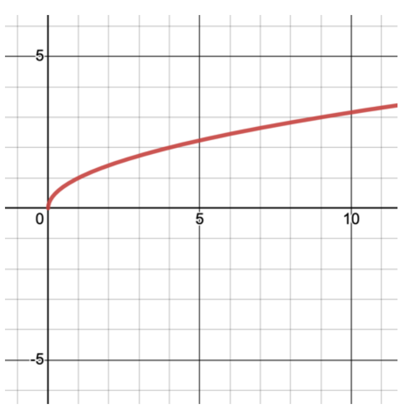
$y = \pm\sqrt{x}$

Domain:

Range:

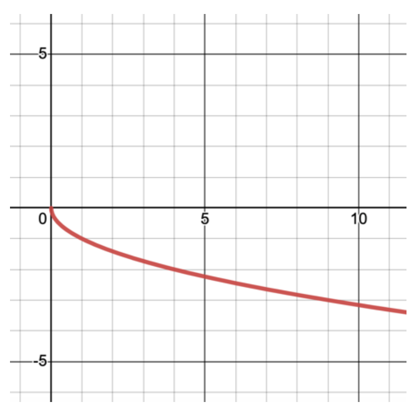
Is this a function?

Example 3: Below are square root equations with a restriction on the range. What would the restriction on the range have to be to make each of these a function?



$y = +\sqrt{x}$

Range:



$y = -\sqrt{x}$

Range:

Quadratic equations without a linear term are the easiest to solve because there is only one x to solve for in the equation. To solve for the variable, isolate the x on one side. This means at one point you will have to take the square root of both sides of the equation.

Example 4: Solve the quadratic equation $x^2 + 10 = 46$. Find the values of x. There will be two solutions.

Example 5: Solve the quadratic equation $x^2 = 5$. Find the exact solution and the decimal approximation.

Example 6: Solve the quadratic equation $(x + 2)^2 = 16$. (Hint: You will have to take the square root of both sides of the equation first because of the parenthesis. You cannot subtract the 2 first because it is inside parenthesis.)

A Russian mathematician, Nikolai Luzin, said the concept of function is one of the most fundamental of all mathematics. He said it did not come suddenly, but over the course of 200 years of debate about a vibrating string: this vibrating string to represent y as an output given the input x. It has always been closely connected to scientific studies.

The actual notation $f(x)$ was first introduced by Leonhard Euler in 1734. Functions often get confused with formulas, algorithms, or equations by definition. All quadratics, by definition, are functions. Their graphs are always parabolas and they always pass the Vertical-Line Test. Remember, quadratic functions come in many forms.

Example 7: Convert the vertex form quadratic equation to standard form by expanding the binomial square and combining like terms.

$$y = (x + 1)^2 - 4$$

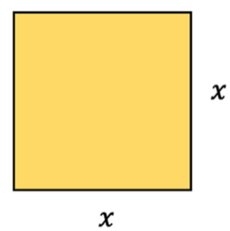
Example 8: Convert the vertex form quadratic equation to standard form by expanding the binomial using the Distributive Property and combining like terms.

$$y = -3(x - 2)^2 + 8$$

Section 6.2 Quadratics and Area Problems

Looking Back 6.2

We have seen that when we graph quadratic equations in algebraic form on the xy-coordinate plane, where values are input for x in the equation and output values for y are calculated, we always get the shape of a parabola. We have learned that the geometric representation of a number multiplied by itself yields a square because the length and the width are the same. That means the geometric representation of the parent function $y = x^2$ is a square with a side length of x and an area of x^2.



$A = x^2$

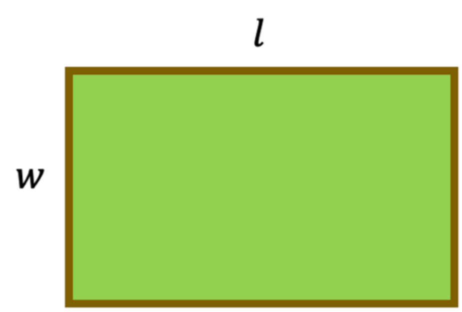
A quadratic equation can be represented geometrically: as a perfect square if the factors representing the side length and side width are the same; as a rectangle if the side length and side width are not the same. The factored form then represents the length and width of the sides of the quadrilateral.

In the previous module, we used multiple representations for polynomial equations. In this section, we will look at some examples to help us make a connection between the algebraic representation and the geometric representation of a quadratic equation, which are polynomial equations of the second degree. The tabular, graphical, and geometric representations are multiple ways of investigating these equations.

Looking Ahead 6.2

Example 1: La'Daisha buys a new puppy and wants to have a pen built for him in the backyard. The local lumber yard has 16 linear-feet of fence on sale. La'Daisha wants the pen to have the maximum area for her puppy to run around. What should the length of the sides of the pen be if the perimeter is 16 feet? Find the width in terms of the length first because the length is known.

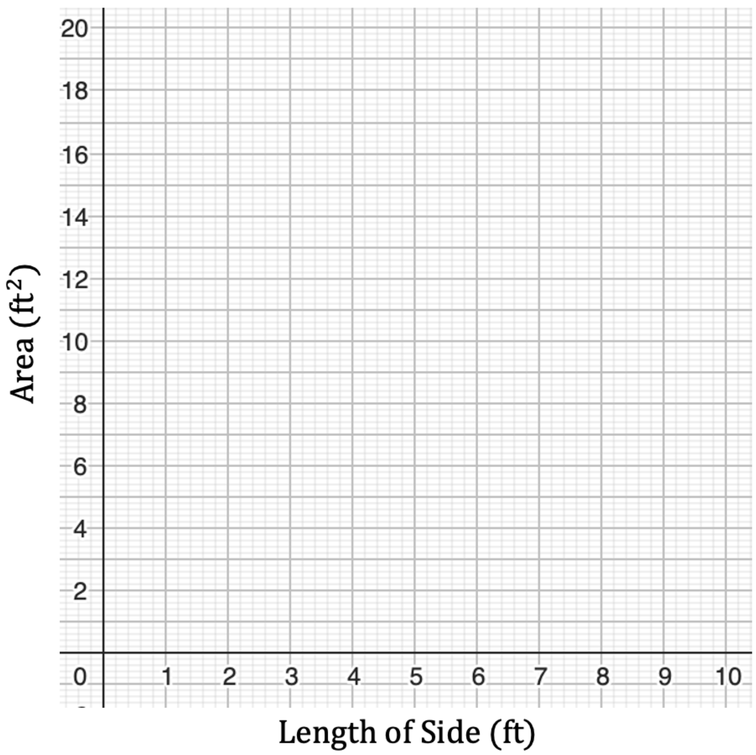
Length (l)	**Width** $(w) = (8 - l)$	**Area** $= l \cdot w$
0		
1		
2		
3		
4		
5		
6		
7		
8		



$P = 2l + 2w$ or $2(l + w)$

$A = l \cdot w$

Example 2: Looking at the table in Example 1, as the length increases, the area increases until the maximum area of 16 ft^2 is reached. Then, as the length of the sides continues to increase, the area decreases. When the length is 0, the area is 0. When the length is 8 ft, the area is 0. These are the two x-intercepts on the graph: $(0, 0)$ and $(8, 0)$. If x is the length of the side and y is the area of the pen, what will the graph look like?



Example 3: Analyze the parabola from Example 2, relating it to the problem in Example 1, to fill in the blanks below.

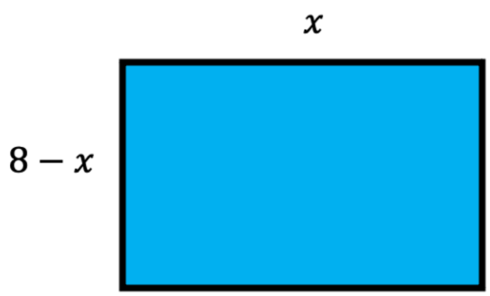
a) The vertex is the ____________________ point on the graph. It represents the _____________________ area.

b) The line of symmetry is ____________________ and crosses the middle of the graph at the point _____________ which is halfway between the x-intercepts.

c) The parabola opens ________________________.

d) The two x-intercepts are when the area is 0. The length is either ___________ or __________.

Example 4:	Find the quadratic equation that represents the area of the quadrilateral. Write it in standard form.



$\text{Area} = l \cdot w$

$\text{Area} =$ ________ $\cdot$ ________

Section 6.3 The Zero-Product Property

Looking Back 6.3

In the previous Practice Problems section, the length was not always the longest side and the width was not always the shortest side. Length and width were interchangeable. Going back to the formula for perimeter, $P = 2(l + w)$, we can see…

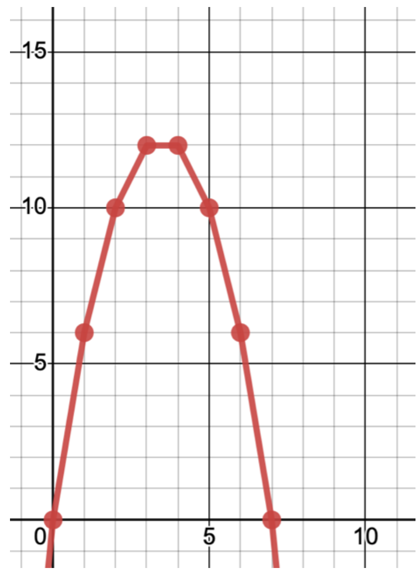
$$\frac{P}{2} = l + w \Rightarrow \frac{P}{2} - l = w$$

Or

$$\frac{P}{2} = l + w \Rightarrow \frac{P}{2} - w = l$$

The symbol $\Rightarrow$ means "Therefore." Length and width seem to be interchangeable because the graph is symmetric. The table and graph below represent a puppy pen with a perimeter of 14-feet.

Width (w)	**Length** $(l) = (7 - w)$	**Area** $= l \cdot w$
0	7	0
1	6	6
2	5	10
3	4	12
4	3	12
5	2	10
6	1	6
7	0	0



As the width increased, the length decreased; therefore, one affects the other. The area increases and then decreases.

If we let x represent the width (input) and y represent the area (output), then we see the same relationship in the graph of the data. Quadratic relationships seem to have an underlying multiplicative relationship.

Looking Ahead 6.3

Remember that in the table the x-intercepts are located where the value of y is zero. Similarly, on the table, the y-intercepts are located where the value of x is 0. If a point is located on the y-axis, the movement from the origin is either up or down. If there is vertical movement from the origin, there is no x-intercept. So, the value of x is 0 at the y-intercept.

Now, on the graph, we see that the x-intercepts are where the quadratic equation crosses the x-axis. Also, on the graph above, the x-intercepts are $(0, 0)$ and $(7, 0)$. Notice that the maximum point of the graph is in the middle of the x-intercepts. The graph goes up before $x = 3.5$ and the graph goes down after $x = 3.5$. That means that $x = 3.5$ is the axis of symmetry. Therefore, we can find the x-intercepts of our quadratic equation by making $y = 0$. Once we graph these points, we can find the axis of symmetry, which is halfway between the two x-intercepts. That means the value of x at the vertex is also 3.5, the same as the axis of symmetry. The maximum point is halfway between the width of 3 feet and 4 feet, so it is 3.5 feet. The length and width are the same so the area is $(3.5) \cdot (3.5)$ or $12.25 ft.^2$

Example 1: In quadratic relationships, the y-value is the product of two linear factors. These factors are in the form $(ax + c)(bx + d)$ in which $a \neq 0$ and $b \neq 0$. The variables a, b, c, and d represent real numbers. In the standard form, the quadratic equation is $ax^2 + bx + c$; there is no d. The constants a, b, and c in the standard form are not the same as the constants a, b, c, and d in the factored form. The area problem above is represented by the equation $A = w(7 - w)$. Substitute x for width and y for area and find the two linear factors in terms of a, b, c, and d.

Example 2: Because x represents the width and $7 - x$ represents the length of the area above, use the Zero-Product Property to find the x-intercepts of the quadratic equation with an area that can be geometrically represented by a rectangle.

Substituting 0 for y in the equation $(ax + c)(bx + d) = y$, we can find the x-intercepts using the Zero-Product Property. If $(ax + c)(bx + d) = 0$, then either $ax + c = 0$ or $bx + d = 0$.

Example 3: Find the x-intercepts from the linear factors of the quadratic equation $(x+3)(2x-1)=y$ and check your solutions.

Example 4: Expand the linear factors in Example 2 and 3 to write the quadratic equation in standard form.

Section 6.4 Finding the Vertex of a Quadratic Equation

Looking Back 6.4

In the previous section, using the Zero-Product Property to solve a quadratic equation helped identify the x-intercepts of the equation. This occurs when $f(x) = 0$. The x-intercepts of the graph of $y = f(x)$ are also called the *zeroes* or *roots* or *solutions* of the quadratic equation. The zeroes of $y = (x - 3)(x + 4)$ are $x = 3$ and $x = -4$. These zeroes are the numbers that make $f(x)$ (or y) equal 0. They are the x-intercepts and on the graph, they are the ordered pairs $(-4, 0)$ and $(3, 0)$. In each case, the value of y is 0.

The Zero-Product Property of multiplication states that for all real numbers a and b…

If $ab = 0$, then $a = 0$ or $b = 0$ or both a and b equal 0.

Also, if $a = 0$ or $b = 0$ or a and b equal 0, then $ab = 0$.

We have used a lot of different terms when discussing quadratics. Before we go on, let us review some here.

The x-intercepts can be called the following:

"The zeroes of the function;" "The roots of the function;" "The solutions of the equation"

The independent variable (input) is x and the dependent variable (output) is y (or $f(x)$ using function notation).

Looking Ahead 6.4

Example 1: In the equation described above, the zeroes of $y = (x - 3)(x + 4)$ were 3 and -4. Now, find the x-value of the vertex.

The x-value of the vertex is the point halfway between the x-intercepts and lies on the axis of symmetry.

Example 2: Find the y-value of the vertex of $y = (x - 3)(x + 4)$. Substitute the x-value that we found for the vertex in Example 1 into the equation and solve for y.

$$y = (x - 3)(x + 4)$$

Substitute $-\frac{1}{2}$ for x and solve for y.

The ordered pair of the vertex is ____________________.

The vertex should be a ____________________ because the parabola opens ______________________ because

a __________ 0.

Example 3: Change the factored form of the quadratic equation $y = (x - 3)(x + 4)$ to standard form. Then substitute in the value for x found in Example 1. Solve for y and see if you get the same y-value for the vertex as you did in Example 2.

Again, the vertex is ____________________ because the equations are the same, just in two different forms. Their graphs would be exactly alike.

Example 4: Complete the following steps, which are the steps for finding the vertex of a quadratic equation.

Step 1: Find the _____________________(x-intercepts) using the __________-___________________ Property.

Step 2: Find the distance ____________ between the x-intecepts. This is the x-value of the vertex (x_v).

Step 3: Substitute x_v into the quadratic equation in ____________________ form or ___________________ form. This will give you the y-value of the vertex (y_v).

Example 5: Find the zeroes of $f(x) = (x + 2)(x - 5)$ and then find the vertex using the steps from Example 1.

Section 6.5 Horizontal Shifts in Quadratic Equations

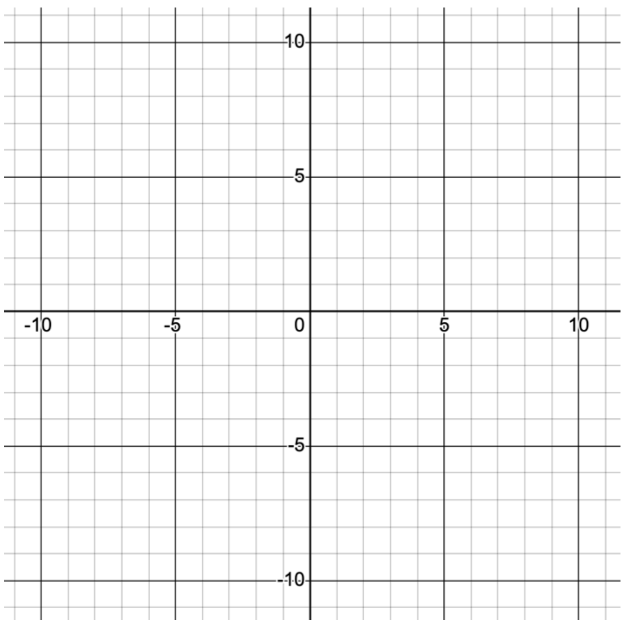
Looking Back 6.5

The $ax^2 + bx + c = y$ form of a quadratic equation is called the standard form. The standard form is the way an equation typically written. The quadratic term is first, followed by the linear term, followed by the constant term.

The $(ax + c)(bx + d)$ form of a quadratic equation is called the factored form. As we said earlier, a, b, c, and d represent numbers or parameters in this specific equation, but represent four different numbers or parameters in another equation or other equations of other forms. That is why they are listed as variables here.

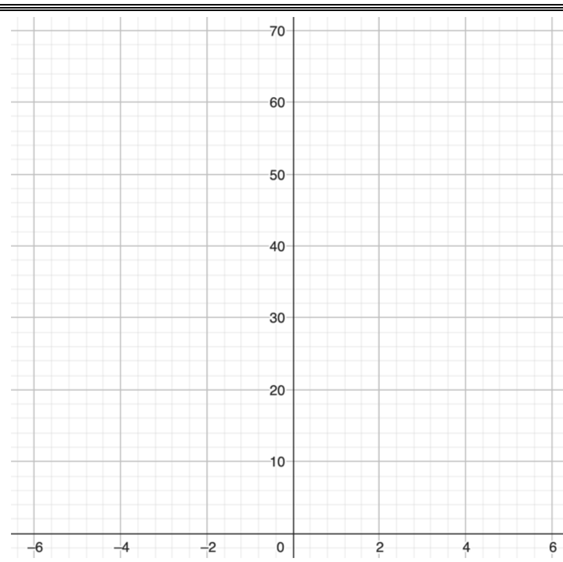
We also have a form called the vertex form: $a(x - h)^2 + k = 0$. In the next few sections, we will investigate the vertex form and methods to convert standard form to vertex form.

Example 1: The parent function of a quadratic equation is $y = x^2$. Sketch its graph and name the vertex.



Example 2: Complete the table for $y = (x - 3)^2$ and graph it on the coordinate graph.

x	y
-2	
-1	
0	
1	
2	
3	
4	
5	
6	
7	
8	



Example 3: Looking at the graph of Example 2, what are the x-intercept(s) and the vertex of the quadratic equation?

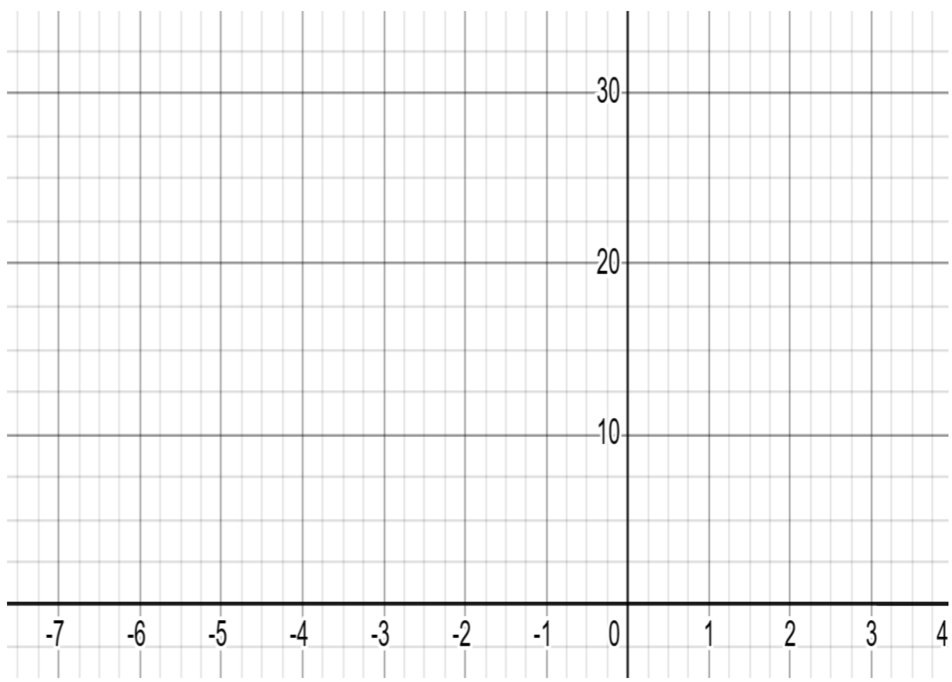
Example 4: Find the x-intercept(s) and vertex from Example 3 algebraically; substitute $y = 0$ and solve for x.

Example 5: Compare the parent function, $y = x^2$ to $y = (x - 3)^2$. How did the graph shift and why do you think it did shift?

Example 6: Look at vertex form of a quadratic equation, which is $a(x-h)^2+k=0$ in which a, h, or k are parameters. Is the value of 3 in $y=(x-3)^2$ the parameter a, h, or k in the graphing form of the equation? What does the value of 3 represent?

Example 7: What do you think the graph of $y=(x+3)^2$ will look like and why? Follow the steps from Examples 1-6.

x	y
-8	
-7	
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	



In the vertex form of the equation $y=a(x-h)^2+k$ in which $a=1$ and $k=0$, h represents a horizontal shift. If h is positive, the graph shifts to the right that number of spaces and if h is negative, the graph shifts to the left that number of spaces. When h is a negative number, $(x-(-h))^2$ becomes $(x+h)^2$.

Section 6.6 Vertical Shifts in Quadratic Equations

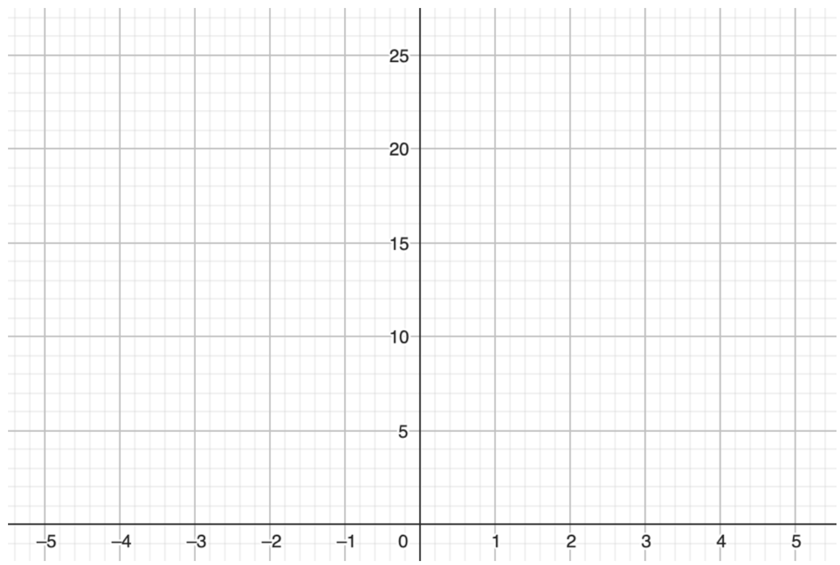
Looking Back 6.6

In the previous section, we saw that when the constant value was inside the parenthesis with the x it caused a horizontal shift of the graph of the parent function. The function $y = a(x-h)^2 + k$ in which $a = 1$ and $k = 0$ is $y = (x-h)^2$. It looks just like the parent function $y = x^2$, but is shifted either to the left or right of the origin on the coordinate plane. The new graph could be in Quadrant I, Quadrant II, or both Quadrant I and II, but it never goes into Quadrant III or Quadrant IV because the shift is only left or right. To end up in Quadrant III or IV, the graph must shift up or down. We call a shift up or down a vertical shift.

Looking Ahead 6.6

Example 1: Complete the table for $y = x^2 + 3$ and graph it. What is the vertex of the graph?

x	y
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	



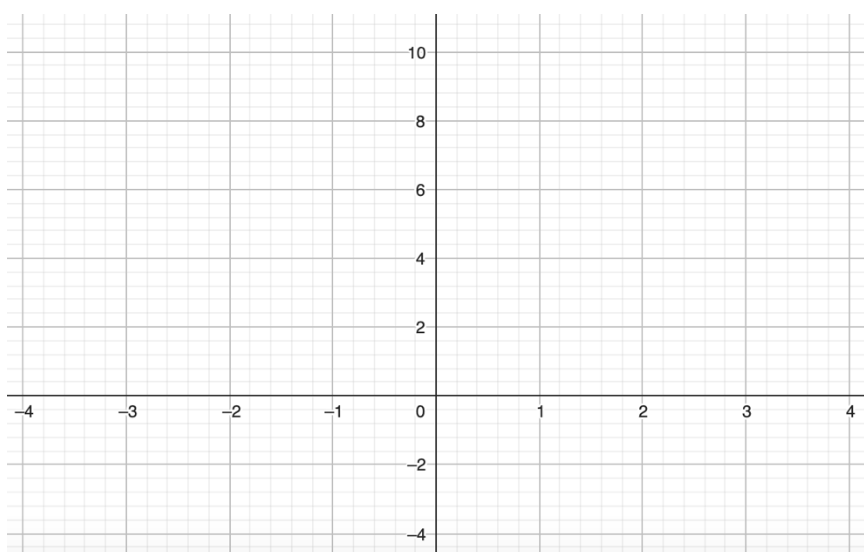
Example 2: Look at the table and graph in Example 1. Find the y-intercept of the graph algebraically.

Example 3: How do the graphs of the equations $y = (x + 3)^2$ and $y = x^2 + 3$ compare?

Example 4: Sketch a graph of what you think the transformation $y = x^2 - 2$ will look like. Complete the table for the transformed function $y = x^2 - 2$ and graph it. What is the vertex of the graph? What is the y-intercept of the graph? How does transformed function compare to the parent function of the graph?

The constant represents the k parameter in the graphing form of a quadratic equation: Is k positive or negative?

x	y
-3	
-2	
-1	
0	
1	
2	
3	



Section 6.7 Vertex (or Graphing) Form

Looking Back 6.7

Now, it makes sense that $y = a(x - h)^2 + k$ is called the vertex form because h and k represent the vertex of the equation. This form is the factored form found by completing the square. We will review this form in the next section.

In this section, we will look at the effect that a has on the graph in terms of opening upward or downward. It also causes a vertical stretch or vertical compression of the graph, which we will explore further in Algebra 2. In this module, we will not investigate where this effect comes from or how to find it; again, we will explore this further in Algebra 2.

Moreover, in this section, we will also practice understanding and using the vertex form. The vertex form is also called the graphing form because we can easily see the vertex in the equation, which makes it easier to graph.

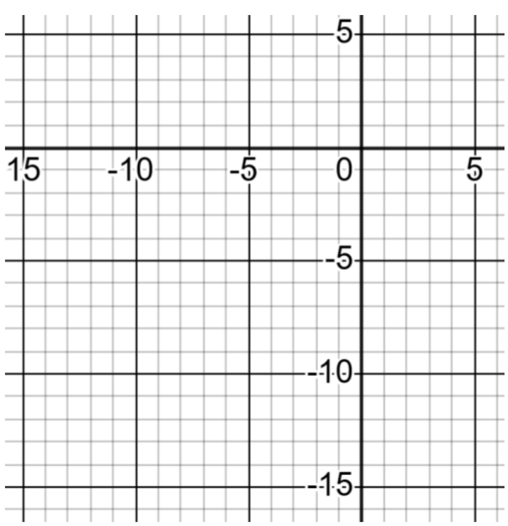
Looking Ahead 6.7

We have learned that when there is no constant for a then $a = 1$. Therefore, we have the equation $y = (x - h)^2 + k$. When a is positive, the graph opens upward and when a is negative, the graph opens downward.

Example 1: Graph the quadratic equations for $y = x^2$, $y = 2x^2$, and $y = \frac{1}{2}x^2$ all on one graph (below). What do you notice?

x	x^2	$2x^2$	$\frac{1}{2}x^2$
-3			
-2			
-1			
0			
1			
2			
3			

Example 2: Graph $y = -3(x + 2)^2 - 3$. What effect does $a = -3$ have on the graph?



x	y
-4	
-3	
-2	
-1	
0	

Example 3: Use $y = a(x+2)^2 - 3$ and one point from Example 2 to solve for a. Try it with another point to see if you get $a = -3$.

Example 4: Substitute $y = 0$ in the vertex form of the quadratic equation and solve for x to find the x-intercepts of the graph.

$$y = (x-6)^2 - 25$$

Section 6.8 Factoring to the Vertex Form

Looking Back 6.8

The quadratic equation $x^2 - 9x + 18 = y$ can be factored into an equation with two linear factors: $(x - 3)(x - 6) = y$. The Zero-Product Property can then be used to find the zeroes of the equation. The zeroes can be used to graph the equation and the parabola that models it. The vertex can also be determined.

The purpose of factoring polynomials is in large part to graph the equation and interpret the data in terms of a real-world problem or situation. Values can always be substituted for x in the equation to calculate values for y. The input and output are then used to create a table from which ordered pairs can be graphed. It is quite difficult and may take a long time to find the critical points, such as the vertex. However, this method of substituting values for x to solve for y does work to graph an equation no matter how simple or complex. We can understand why factoring a polynomial to look for critical points to graph is a preferable method to substitution.

When the method called Completing the Square is used to factor a quadratic equation, then the vertex form is the end result and the vertex is easy to identify. We have learned about Completing the Square and the vertex form when we used the T-charts to factor polynomials. An example would be converting $x^2 - 4x + 8 = y$ to $(x - 2)^2 + 4 = y$.

We factored by completing the square with polynomial blocks and factored algebraically in Module 5. We will begin this section by reviewing the steps that were derived from those investigations.

Looking Ahead 6.8

To factor $y = x^2 - 4x + 8$ into the vertex form, $y = a(x - h)^2 + k$, both k and h must be determined. We can do this by factoring using the Completing the Square method. The steps to do this are listed below.

Make sure the equation is in the form $ax^2 + bx + c = 0$:
Step 1: Rewrite the quadratic equation in the form $ax^2 + bx = -c$
Step 2: Take $\frac{1}{2}b$ and square it
Step 3: Add that number to both sides of the equation to form a perfect square trinomial and simplify it
Step 4: Rewrite the left side of the equation as a binomial square and combine constants on the right side
Step 5: Rewrite the equation by setting it equal to 0
This will give us the form $a(x - h)^2 + k = 0$.

Example 1: Factor $y = x^2 - 4x + 8$ by completing the square.

We write the equation in the form $x^2 - 4x + 8 = 0$:

Rewrite the equation as $ax^2 + bx = -c$:

Take one-half of b:

Square one-half of b:

Add $[\frac{1}{2}(b)]^2$ to each side of the equation:

Factor the left side of the equation as a binomial square and combine constants on the right side of the equation:

Rewrite the equation by setting it equal to 0:

Now, write this quadratic equation in vertex form:

We can see the vertex is $(2, 4)$.

Example 2: Factor $x^2 - 2x + 9 = y$ into vertex form by completing the square.

Rewrite $x^2 - 2x + 9 = 0$:

Take one-half of b:

Square one-half of b:

Add $[\frac{1}{2}(b)]^2$ to each side of the equation:

Factor the left side of the equation as a binomial square and combine constants on the right side of the equation:

Rewrite the equation by setting it equal to 0:

Do you see that when we found one-half of the middle term, we found the constant term of the binomial?

Example 3: Check your solution in Example 2 by expanding your solution to the standard form of the quadratic equation. Is it the same as the quadratic equation in Example 2?

Example 4: Factor $x^2 + 16x - 36 = 0$ into vertex form by completing the square. Name the vertex and tell what the shifts are from the parent function.

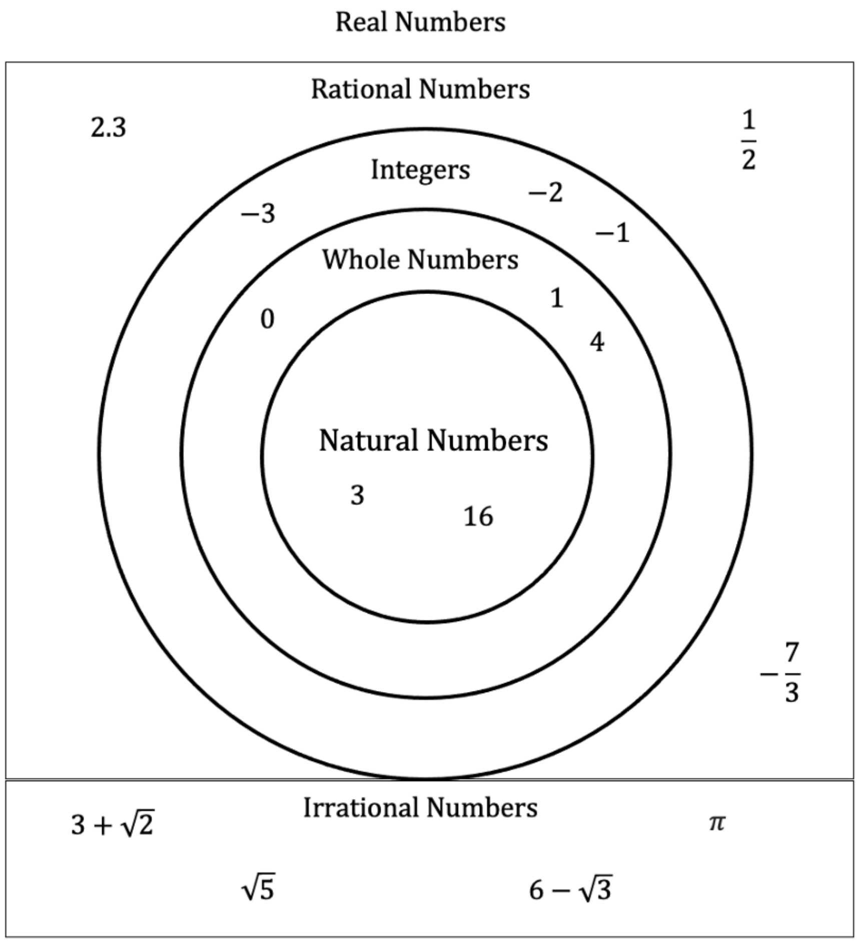
Example 5: Now that Example 4 is written in vertex form, substitute 0 for y and solve for x to find the x-intercepts.

Example 6: Complete the square for $x^2 - 4x - 12 = 0$ to find the vertex form of the equation. Find the vertex and the x-intercepts of the equation.

Section 6.9 Irrational Numbers and Quadratics

Looking Back 6.9

Before we begin investigating irrational numbers, let us review integers and rational numbers using the Venn Diagram below.



Natural numbers are counting numbers.
They are rational numbers.
Whole numbers are the natural numbers and 0.
They are rational numbers.
Integers include whole numbers and the natural counting numbers. They also include negative numbers.
They are rational numbers.
Rational numbers are numbers that can be written as quotients/fractions, $\frac{a}{b}$ in which $b \neq 0$; if the number is a decimal, they will either repeat or end.
Irrational numbers cannot be written as fractions and if they are decimals, they do not repeat or terminate (come to an end).
Rational and irrational numbers are real numbers.

Looking Ahead 6.9

A quadratic equation is a function, so it may have two input values for every output. For example, if $x^2 = 9$, then the input could be 3 or -3. In fact, when we have graphed a quadratic equation and looked at its parabola, we saw that each parabola had a line of symmetry through the vertex. We had two x-values for each y-value, except at the vertex, which is either a minimum or maximum point.

Example 1: In the equation $(x+2)^2 - 10 = y$, find the two values of x that will give a y-value of 39.

These are the rational solutions.

Example 2: In the equation $(x+2)^2 - 10 = y$, find the two values of x that will give a value of 37. Write the solution both in exact form and as a decimal approximation.

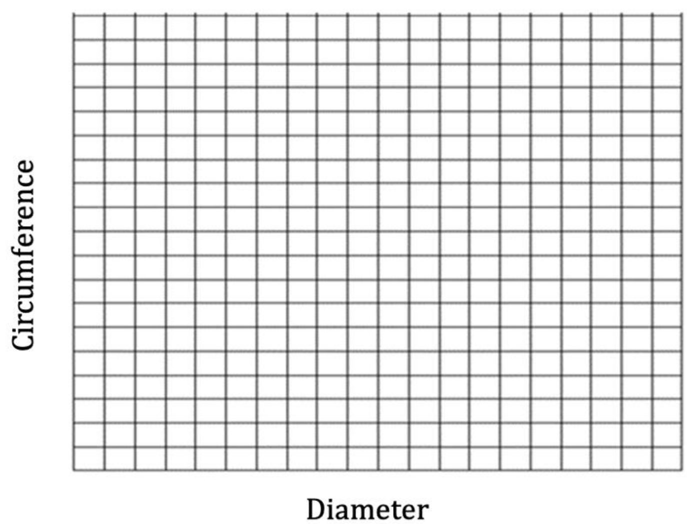
Now, let us look at two more examples: One, linear with additive properties and; Two, quadratic with multiplicative properties, which involve a transcendental irrational number.

The equation $C = \pi d$ is a direct variation in which the circumference, C, of the circle is directly proportional to the diameter, d, of the circle. The circumference, C, is the distance around the circle and the diameter, d, is the distance from one point on a circle to one point on the opposite side of the circle that runs through the center of the circle.

Example 3: Using six tin cans of different sizes and a tape measure, measure the circumference and diameter of each tin can and record them in the table below. If you do not have a tape measure, wrap string around the can given and then unwrap the string and measure the length formerly around the can and measure it on a ruler. It is easiest to measure in millimeters.

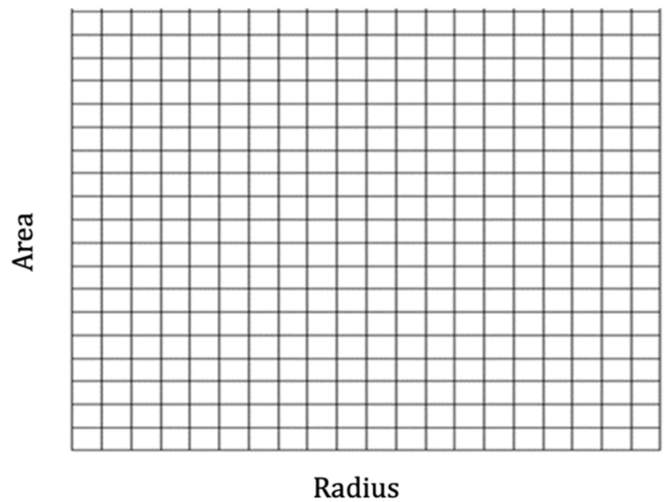
Once you have measured the circumference and diameter of each tin can, divide the circumference by the diameter of each tin can and record them in the table. Add all of these ratios together to find the constant of variation.

c	d	$k = \frac{c}{d}$



Example 4: From Example 3, use the constant of variation, pi, to find the area, $A = \text{pi}\ r^2$, in which the area of the circle is directly proportional to the radius of the circle squared. Complete the table and graph below.

r	$A = \pi r^2$



Section 6.10 Projectile Motion

Looking Back 6.10

Equations in mathematics are often used to model real-world problems or situations. Quadratic equations are used to model projectile motion problems such as the takeoff or flight of a rocket or throwing or hitting a ball.

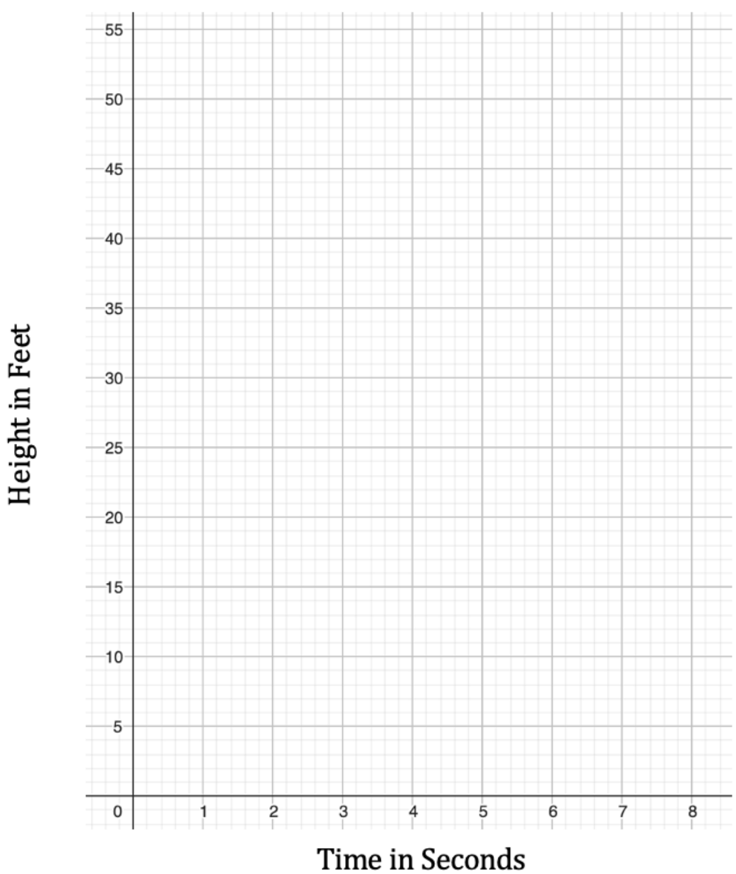
A ball that is hit or thrown has a starting position. It leaves the thrower's hand and decreases in speed until it reaches a maximum height because the forces of gravity are acting upon it. Upon reaching the maximum height, the ball will then return to earth and increase in speed until it is caught or hits the ground.

Gravity causes objects to accelerate downward toward the earth and increase in speed every second. The acceleration due to gravity is 32 ft/sec^2 (32 feet per second squared) or 9.8 m/sec^2 (9.8 meters per second squared).

Looking Ahead 6.10

Example 1: A father and a son are playing baseball in the backyard. The fathers throws a pop-up for his son to catch. The ball leaves the father's hand at 8 ft 6 in and travels up roughly 54 ft (including the 8 ft 6 in) before descending. After 8 seconds, the son catches the ball in his glove, which is extended to 6 ft 2 in.

Draw the graph of this situation. Let x represent the time in seconds (s). Let y represent the height of the ball in feet (ft). The y-intercept (starting height) is 8 ft 6 in. The ball increases in speed until 54 ft and then decreases, stopping at a height of 6 ft 2 in when it is caught by the son. The maximum point is 54 ft: this is the vertex of the parabola. The height of $6'2''$ is the point at which the ball is caught 8 seconds later; the end point is $(8, 6'2'')$. The graph of distance over time for a thrown ball is a parabola. The path of the ball itself goes almost straight up and then down.



It is important to note that this is not the path of the ball, but the path of the height of the ball over time. It is a transformation of the parent function. It shows both horizontal and vertical movement.

Example 2: Suppose the son from Example 1 throws a basketball straight up in the air. The equation that models the height of the ball is $h(t) = -16t^2 + 64t + 7.5$. The table for the equation is shown below.

$\boldsymbol{t}$ (time in seconds)	$\boldsymbol{h(t)}$ (height in feet)
0	7.5
0.2	19.66
0.4	30.54
0.6	40.14
0.8	48.46
1	55.5
1.2	61.26
1.4	65.74
1.6	68.94
1.8	70.86
2	71.5
2.2	70.86
2.4	68.94
2.6	65.74
2.8	61.26
3	55.5
3.2	48.46
3.4	40.14
3.6	30.54
3.8	19.66
4	7.5
4.2	-5.94

What does the first ordered pair, $(0, 7.5)$, represent? What does the notation $h(t)$ mean?

What appears to be the maximum height reached by the ball and after how many seconds does the maximum height occur?

If the son catches the ball at 7.5 feet, after how many seconds does the catch occur?

Why do negative heights not make sense?

To learn more about projectile motion, you can build a water-bottle rocket out of 2-liter soda bottles. You can also use a bicycle tire pump and PVC pipe to build a launcher. The bottle is filled with water, the lid is removed, and the bottle is connected to the launcher. Then it is turned upside down and pumped and the pressure causes the rocket to launch into the air.

After it is built, have two timers time the launch from the start until the water-bottle rocket lands on the ground. Then add the times together to find the average time of the flight (hang time).

To calculate the velocity (speed with direction) of your rocket, you will use the vertical motion formula:

$$h(t) = -16t^2 + v_0 t + s$$

$h(t)$ is the height of the rocket (in feet) at time t
t is time (in seconds)
v_0 is initial velocity (ft./sec.)
s is the initial or starting height of the rocket

Launch the rocket and have two people time the hang time in the air and find the average time of flight (t) and solve for v_0 to find the initial velocity.

Use dimensional analysis (the cancel/keep) method to calculate the velocity of the rocket in miles per hour (mph). Then you will solve for y or $h(t)$ to find the maximum vertical distance or height the rocket reaches. This will be in miles per hour, but you will convert it to feet per second (fps). You will have a chance to do this in the practice problems.

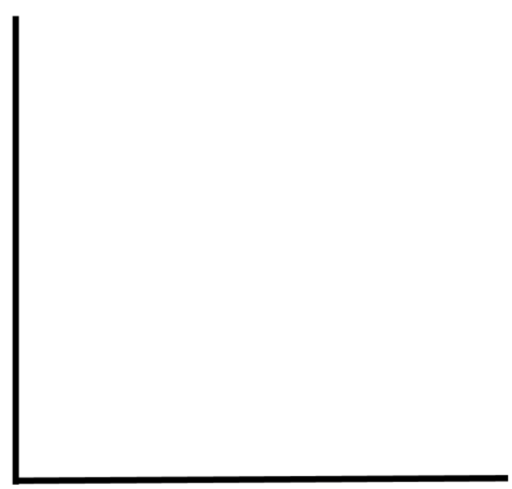
The next few examples will demonstrate how to do the rocket launch calculations.

Example 3: If your rocket is airborne for 4.1 seconds, what is the initial velocity in feet per second? When the rocket lands $h(t) = 0$. Convert the velocity to miles per hour using the cancel/keep method. Hint: 1 mile = 5,280 feet

Example 4: Calculate the velocity of your rocket if it is in the air for 5.4 seconds. Use the formula $h(t) = -16t^2 + v_0 t + s$. Convert the velocity from feet per second to miles per hour.

The vertex of the parabola occurs at the maximum (x_v, y_v) which in this case is halfway through the flight. This is represented by $(t, h(t))$ where t is the time halfway through the flight and $h(t)$ is the maximum height that also occurs halfway through the flight. It is the vertex of the parabolic path.

Example 5: Mr. Swihart builds the "Ace Flyer," and it blasts off with a velocity of 28 miles per hour. Convert this to feet per second. Find the time and height of the "Ace Flyer" when it is at its highest or maximum point. Sketch a graph that models the situation.



Section 6.11 Vertex to Standard Form

Looking Back 6.11

In the previous sections, we have factored quadratic equations using many methods in order to graph them. Factoring by finding the Greatest Common Monomial and factoring by finding linear pairs were first used to factor, and should always be used first if possible. Later, we converted to vertex form by completing the square. It can be difficult to remember the steps if we forget why we are doing what we are doing, but it does always work, even with fractions and decimals as factors (not just integers).

The reverse process will be used in this section: converting vertex form to standard form. Instead of factoring out common terms, the common terms are multiplied through and combined to simplest terms.

A quadratic equation can be converted from vertex form to standard form by expanding the polynomial using multiplication. The standard form is also referred to as the standard form: $ax^2 + bx + c = y$. We will call it the standard form in this section. Converting from other forms to standard form will be the topic of discussion in this section.

It has been stated that quadratic equations are used to model projectile motion such as in rocket launches and rocket flights and so on. Problems in engineering and science are often modeled by quadratic equations. Sometimes, a factored equation must be converted from standard form so that the equation for projectile motion can be found.

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Aristotle was a Greek philosopher and scientist of antiquity. He was much interested in projectile motion and falling objects, but incorrectly believed that heavier objects fall faster to the ground than lighter objects. Galileo, another scientist, but who lived in the 1500s, challenged Aristotle's incorrect assumption.

While observing hailstones during a storm, Galileo noticed the stones fell to the earth and hit the ground at the same time no matter what their size. To test his theory, Galileo went to the top of the Leaning Tower of Pisa and dropped two objects of different weights at the same time. The objects hit the ground at the same time.

However, if we were to drop a feather and an elephant from the Leaning Tower of Pisa at the same time, the elephant would land first. This is because of the air resistance of the very light feather. Outside of those affected by air resistance, objects of any size will land at the same time.

During his day, Galileo was criticized for his theories, even exiled eventually. However, his theories proved to be true. This is a good reminder to challenge popular thinking, and not simply to believe what you are told.

Looking Ahead 6.11

Example 1:	Convert the vertex form of the quadratic equation to the standard form.

$$y = -3(x + 4)^2 - 6$$

You can also write the factored form of a quadratic equation given the zeroes. If you know the zeroes of a quadratic equation, they can be written in factored form because they are linear factors. (The value of a, the coefficient of x^2, the quadratic term, must equal 1.) Once the equation is in factored form, it can be converted to standard form.

Example 2:	Write the quadratic equation in standard form given the zeroes. Let $a = 1$.

$x = 6$ or $x = 4$

Example 3:	Write the quadratic equation in standard form given the zeroes. Let $a = 1$.

$x = 5$ or $x = -7$

Example 4:	Write the quadratic equation in standard form given the zeroes. Let $a = -2$.

$x = -3$ or $x = -7$

Section 6.12 The Quadratic Formula

Looking Back 6.12

We have investigated several forms of the quadratic equation. They can be converted from one form to another.

From standard form to vertex form:

$-4.25x^2 - 17x + 42 = y$ to $-4.25(x-2)^2 + 25 = y$

From standard form to factored form:

$x^2 - 25x = y$ to $x(x-25) = y$

From perfect square trinomial form to binomial square form:

$x^2 + 4x + 4 = y$ to $(x+2)^2 = y$

Quadratic equations cannot always be factored using these methods. However, one method, Completing the Square, can be used to factor any quadratic equation. There is another method that can be used to solve for x and works for any quadratic equation: It is called the Quadratic Formula. We will investigate the Quadratic Formula in this section.

Finding square roots can be used to solve for x in a quadratic equation that has no middle (linear) term. These equations have only the quadratic term (ax^2) and constant term (c), so reverse thinking may be used to solve for x. These terms, ax^2 and c, will be reviewed before exploring the Quadratic Formula because it also involves finding square roots.

Example 1:	Solve for x in the equation $x^2 - 13 = 0$.

Example 2:	Solve for x in the equation $(x+5)^2 = 0$.

Some quadratics cannot be factored or solved with the methods previously learned; they might be very difficult to factor. For example, $y = 0.1x^2 + \frac{49}{17}x - \frac{3}{8}$ is a much more complicated equation to factor using any of these methods, including Completing the Square.

The Quadratic Formula is a method that can be used to solve any quadratic equation in the standard form: $ax^2 + bx + c = 0$. The values of a, b, and c need to be identified in order to use the formula.

The parameters a, b, and c are numbers that name that equation in which: a is the coefficient of the quadratic term; b is the coefficient of the linear term; c is the constant.

Looking Ahead 6.12

Example 3:	Derive the Quadratic Formula. To do this, we begin with $ax^2 + bx + c = 0$ and solve for x.

First, divide the terms by the coefficient of x^2.

$$ax^2 + bx + c = 0$$

$$\frac{a}{a}x^2 + \frac{b}{a}x + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Now, use Completing the Square to get the equation in factored form. Remember, when completing the square, the coefficient of the middle term (b) is found by multiplying the first term and second term of the binomial square, and doubling the product. The binomial square is $(x + ?)^2$.

Use the algorithm to complete the square.

Working backwards, find half of the coefficient of the middle (linear) term:

$$\frac{1}{2}\left(\frac{b}{a}\right) = \frac{b}{2a}$$

Substitute the middle (linear) term in for the question mark (?) above:

$$(x + \frac{b}{2a})^2$$

Expand the binomial square:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = 0$$

What needs to be done to $\frac{c}{a}$ in order to get $\frac{b^2}{4a^2}$?

$$\frac{c}{a} + ? = \frac{b^2}{4a^2}$$

$$? = \frac{b^2}{4a^2} - \frac{c}{a}$$

Find the common denominator:

$$? = \frac{b^2}{4a^2} - \frac{c}{a}\left(\frac{4a}{4a}\right) = \frac{b^2 - 4ac}{4a^2}$$

Substitute these values to complete the square and solve for x:

$$(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\sqrt{(x + \frac{b}{2a})^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Quadratic Formula:

Given the standard form of a quadratic equation, $ax^2 + bx + c = 0$, the zeroes of the equation are given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 4: Solve $9x^2 + 9x = 0$ using the Quadratic Formula.

Example 5: Solve $x^2 - 3x - 4 = 0$ using the Quadratic Formula.

As was stated previously in this module, a quadratic equation can have zero, one, or two solutions. Solving the radical part of the Quadratic Formula determines the number of solutions. The part of the formula under the radical sign, $b^2 - 4ac$, is called the *discriminant*.

If the discriminant is negative, there are no real solutions because the square root of a negative number is an imaginary number. Imaginary numbers will be studied further in *Algebra 2*. There is no reason for using the Quadratic Formula to solve for x if the discriminant is negative (for now).

If the discriminant is positive, there are two solutions. The two solutions can be found using the Quadratic Formula.

If the discriminant is zero, there is one solution. This is called a *double root*.

In the Quadratic Formula, $x = \frac{-b \pm \sqrt{0}}{2a}$ simplifies to $x = \frac{-b}{2a}$; this is the x-coordinate of the vertex of the quadratic equation.

Finding the discriminant helps to determine whether or not the Quadratic Formula needs to be used to solve for x, and how many real number solutions you will have: zero, one, or two solutions.

Example 6: How many solutions does the quadratic equation $x^2 + x + 4 = 0$ have? If it has real number solutions, solve it using the Quadratic Formula.

Example 7: How many solutions does the quadratic equation $x^2 + x - 5 = 0$ have? If it has real number solutions, solve it using the Quadratic Formula. Give your solution in exact form and decimal form.

Example 8: How many solutions does the quadratic equation $x^2 - 2x + 1 = 0$ have? If it has real number solutions, solve it using the Quadratic Formula.

The discriminant can save time if there is no solution. It also divides the work of the Quadratic Formula into two parts, which makes it easier to use.

Example 9: In science class, Mr. Swiharts' students built the Ace Flyer Model Water Bottle Rocket. The rocket was launched from 1 foot above the ground with an initial velocity of 64 feet per second. It cleared a flagpole and landed in a tree 59 feet above the ground. Use the equation $h(t) = -16t^2 + v_0t + h_0$ in which v_0 is the initial velocity of the rocket, h_0 is the initial height of the rocket, and t is time.

Section 6.13 Using Functions for Graphic Design

Looking Back 6.13

We have learned about several equations: linear, absolute value, direct variation, inverse variation, and now quadratics. All these functions can be used to model lines and designs in God's universe.

In this section, we will learn how to do model lines and designs, but we will investigate one more equation: that which makes a circle.

The equation for a circle is $x^2 + y^2 = r^2$ in which r is the radius of the circle. The equation determines the diameter (double the radius), which determines the size of the circle.

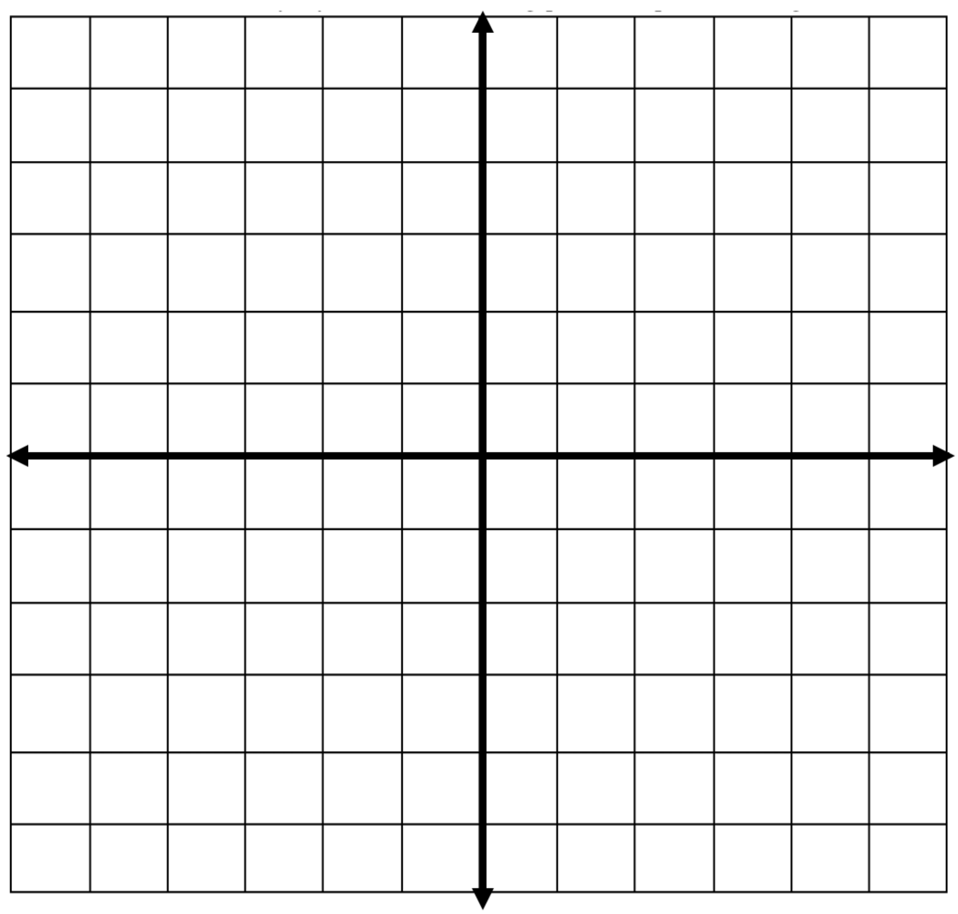
Example 1:	Find the equation of a circle with a radius of 6 units. Solve for y in terms of x.

Example 2:	Shift the circle in Example 1 right 4 units and down 2 units so the center of the circle is $(4, -2)$.

Example 3: Restrict the domain of the circle from Example 2 so only the left half of the upper part of the circle shows.

Looking Ahead 6.13

We will use graph paper to draw an emoticon. Then we will find the lines of the equation that will draw the picture along with the domain restrictions that determine the size of each line. Finally, we will input the equations into the calculator and modify any lines so there are no gaps or overlaps in their design.



Name ________________________________ Picture ________________________________

No.	Function	Equation	Domain