Module 2: Systems of Equations and Inequalities

Section 2.1 Graphs and Solutions

Looking Back 2.1

Now that we have thoroughly explored, investigated, and analyzed linear equations, it is time to apply all that we have learned to linear inequalities. When working with inequalities, it is possible to have more than one solution. If $x > 3$ (read: "x is greater than three"), some possible solutions are: 4, 5, 7.2, $8\frac{1}{2}$, $9.\overline{3}$, etc., or any number larger than 3; however, the number cannot be 3 or any number less than 3. The solutions can be graphed on a number line to show all the possible solutions.

The open circle shows that 3 is not included, but only numbers larger than 3, which are represented by the shaded line. If the circle is closed, then 3 is a possible solution for x and the inequality would be $x \geq 3$ (read: "x is greater than or equal to three").

Think about an inequality being written $3 \leq x$. That is the same as the graph above. If 3 is less than or equal to x, then x must be greater than or equal to 3.

Are the inequalities below true or false?

$x > 2$ or $2 < x$

$x \leq -2$ or $-2 \geq x$

They are both true. It is just two different ways of saying the same thing.

Looking Ahead 2.1

Solutions to inequalities can be determined by evaluating the inequality. This is done just like evaluating expressions. The value of the variable is substituted into the inequality to check and see if the number is a solution.

Example 1:	Tell which numbers are solutions for the inequality below.

$$4 - 7n > 20$$

a) $n = -6$ b) $n = 5$ c) $n = 0$

Example 2:	Substitute the values for the variable in the inequality below to see which ones are solutions.

$$-10 \le t$$

a) 1.7 b) $-4\frac{4}{5}$ c) $\frac{12}{7}$

Section 2.2 Solving Multi-Step Equations

Looking Back 2.2

In Module 1, we simplified algebraic expressions by combining terms in grouping symbols such as parenthesis, ratios, or exponents, then completed any addition and/or subtraction. To take the inverse of an equation, we did the above operations in reverse order to solve for an unknown variable. We found a specific solution for a specific equation.

In the previous section, we solved inequalities involving one step. This section involves multiple steps to solve inequalities. These should be solved exactly like equations in which the variable is isolated on one side of the equation (as when we used balance scales). However, because of the inequality symbol, we will get more than one solution. There is one exception to doing exactly the same steps for equations and inequalities. It occurs during the last step, when we are "undoing" multiplication with division that involves negatives or opposites.

For the equation below, the solution is -4.

$$-3x = 12$$
$$\frac{-3x}{-3} = \frac{12}{-3}$$
$$x = -4$$

Check:
$$-3(4) = 12$$
$$-12 = -12$$
✔

However, if we solve the inequality $-3x < 12$ like we solve an equation, we get the following solution:

$$-3x < 12$$
$$\frac{-3}{-3}x < \frac{12}{-3}$$
$$x < -4$$

Then let us substitute -5, which is less than -4, into the equality for x and see if it works.

This is because multiplying a negative by a number is like taking the opposite of the number. Therefore, the sign must also be changed to its opposite, so less than must become greater than. If 2 is a real number and $-x < 2$, then $x > -2$. When we divide by a negative number, the inequality sign must be changed to the opposite direction, so $>$ (greater than) becomes $<$ (less than) and $<$ (less than) becomes $>$ (greater than). This is also true for multiplying by a negative number.

$$-3x < 12$$
$$\frac{-3}{-3}x > \frac{12}{-3}$$
$$x > -4$$

Because -3 is greater than -4, we can check our solution to see if it works.

What are some other solutions you can try? (Because there are infinite solutions, we cannot write them all, but we can view them partially on the number line below.)

Looking Ahead 2.2

Example 1: The student council collected $\$414.00$ from fundraisers. They want to buy T-shirts for all 27 members of the student council. What is the most they can spend on each T-shirt?

Let $c = \text{the cost of each T} - \text{shirt.}$

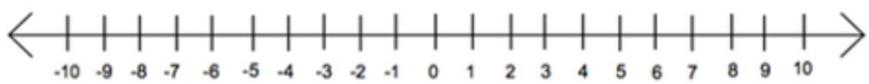
We know that the $(\text{Cost of Tshirts}) \times (\text{Number of Tshirts}) \leq \text{Total Cost.}$

Solve for c:

The words "at most" imply that the cost must be equal to or less than that value.

Example 2: Solve for m in the inequalities below and given the solution to m, graph it.

a) $\frac{m}{2} > 4.1$



b) $2m + 6 \leq 14$

c) $-2(4 + m) \geq m - 3$

Below is the summary of the inequality properties used in this section:

Multiplication Property of Inequality

For every real number a, b, and c when $c > 0$

…

if $a > b$, then $a \cdot c > b \cdot c$

and

if $a < b$, then $a \cdot c < b \cdot c$

For every real number a, b, and c when $c < 0$

…

if $a > b$, then $a \cdot c < b \cdot c$

and

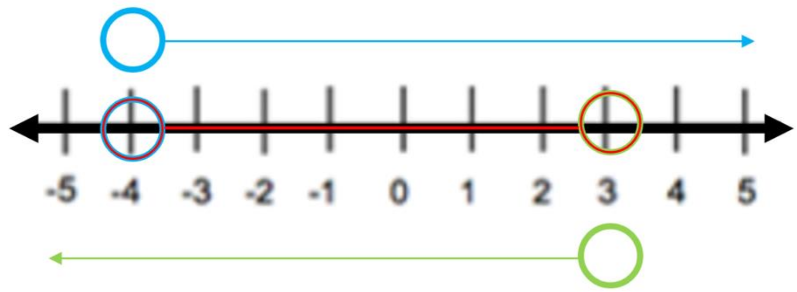
if $a < b$, then $a \cdot c > b \cdot c$

These properties hold true for ≤ and ≥ as well (and for division).

Section 2.3 Solving Compound Inequalities

Looking Back 2.3

Inequalities can be solved much like equations, by isolating the variable on one side and seeing what it is greater than or less than on the other side. The symbol simply changes from $=$ (equal to) to $\leq$ (less than or equal to) or $\geq$ (greater than or equal to), or $<$ (less than) or $>$ (greater than). If a number is less than some numbers in an inequality, and greater than other numbers, then it is any number between at least two other numbers. If we graph $-4 < m < 3$, then m is between -4 and 3, but not equal to either of them. If we write it as two inequalities, the inequalities would be $-4 < m$ and $m < 3$. If they are graphed above and below the number line (as below), the solution graph (red) is the intersection of the two other graphs (blue and green).



So, m is any number between -4 and 3, but does not include -4 and 3. This is called a compound inequality because it is one thing "and" another thing at the same time. We could say it is the intersection of both of the inequalities.

Now, if we have $m < -4$ or $m \geq 3$ then one or the other is true at any given time, but both do not have to be true at the same time. The graph goes in both directions on the number line.

This is also a compound inequality. Compound inequalities use the word "and" or "or" when describing the inequality.

Looking Ahead 2.2

To solve an inequality using the word "and," write the compound inequality as two inequalities. The expression $-2 \leq r - 3 \leq 5$ can be written: "$-2 \leq r - 3$ *and* $r - 3 \leq 5$." The variable r is then found in each inequality. Once found, the two inequalities may be written as one again:

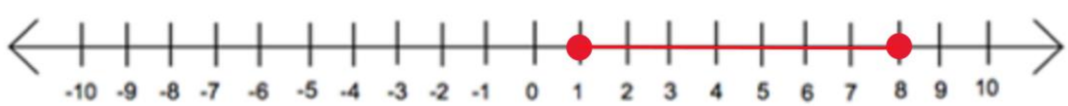
$$-2 \leq r - 3 \qquad \text{and} \qquad r - 3 \leq 5$$

$$+3 \qquad +3 \qquad\qquad\qquad +3 \quad +3$$

$$1 \leq r \qquad \text{and} \qquad r \leq 8$$

$$1 \leq r \leq 8$$

This can be written as one compound inequality and graphed as such: $1 \leq r \leq 8$.



Example 1: The best swimming temperature for a swimming pool is when the average is between $78°$ F and $80°$ F. The temperature of a specific pool was taken and read: "81.2 F." Three hours later, the temperature of this specific pool was taken again and read: "76.3 F." What must the final temperature reading be for the average pool temperature of this specific pool to be between $78°$ F and $80°$ F?

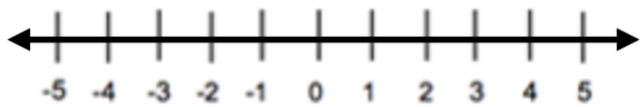
Example 2: In science, you will learn how to find the total force of an object using the following equation:

$$\text{Total Force} = (\text{mass})(\text{acceleration})$$

If a man is pushing his $1{,}500$ kg. car to the nearest gas station to the east, between what amount of force in Newtons must the man push to get the car to move between 0.03 meters per second squared and 0.05 meters per second squared?

Example 3: Solve the compound inequality and graph the solution that follows.

$$14.4m - 8.2 \leq 37.88 \quad \text{or} \quad -6.3m + 4.1 < 17.33$$



When the entire number line is shaded, the solutions is all real numbers.

Summary

For inequalities that use the word “or,” keep the word “or” and graph both solutions, the graph being the union of the two sets.

For compound inequalities that have the variable in between two values, split the inequality into two inequalities using the word “and,” solve for the variable in each, and combine it again in the end for one compound inequality.

The overlap of the separate solutions is the intersection of the two sets.

If there is not overlap for the intersection, then there is no solution.

Section 2.4 Absolute Value Equations

Looking Back 2.4

When something is absolute, it is free of imperfection. In scientific terms, the something has no imperfections in its mixture qualifications. Absolute is also the fundamental units of length, mass, or time. In terms of temperature, absolute zero on the scale corresponds to a complete absence of heat, which is $-273.16^{\circ}\mathrm{C}$.

When we say that God is absolutely supreme, we are saying that His supremacy is perfect. He is perfect, which means there is an absence of any imperfection.

In John 14:6, Jesus says: "*I am the way, the truth, and the life; no one comes to the Father, but through me.*" God is declaring an absolute truth; there is no other way to God except through Christ.

In Revelation 1:8, 21:6, and 22:13, the Lord says: "*I am the alpha and the omega.*" That means He is the first and the last, the beginning and the end. That is an absolute truth.

An absolute value in mathematics means the value of the number. In *Pre-Algebra*, we learned it is the distance from zero because zero is the point of reference. The absolute value of 4 is 4; it is 4 units from zero ($|4| = 4$). The absolute value of a -4 is also 4; it is 4 units from zero ($|-4| = 4$).

Therefore, if we simplify…

$$|2 - 6| = |2 + (-6)| = |-4| = 4$$

Going back to the previous definition of absolute value, there are two numbers representing the absolute value of a given number.

Example 1:	Solve for the variable x in $|x| - 10 = -4$. Be sure to check your solution.

Example 2:	Solve for the variable m in $2|m| + 3 = 21$.

Example 3: If $|n + 3| = 6$, what are the two solutions for n?

The rule for solving absolute value equations is that when $|n| = b$, there are two solutions, either $n = b$ or $n = -b$.

Example 4: Solve for the variable in the equation below. What are the two solutions?

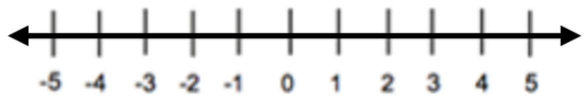
$$|7b + 3| = 17$$

Section 2.5 Absolute Value Inequalities

Looking Back 2.5

We have previously solved inequalities and absolute value equations. In this section, we will put the two together to solve absolute value inequalities. Absolute value inequalities, like regular inequalities, have the following signs: $>$ (greater than), $<$ (less than), $\geq$ (greater than or equal to), or $\leq$ (less than or equal to). The trick to absolute value inequalities is to understand what the absolute value looks like. Let us start with $<$ (less than).

We learned that if $|t| = 5$, then $t = 5$ or $t = -5$. Both are 5 units from 0. What if $|t| < 5$? Think about the values less than five (five spaces from 0) with respect to 0.



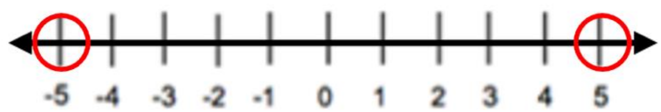
One way to think of this is as if a goat is attached to stake at 0. The farthest the goat can go right is to 5 and the farthest he can go left is to -5. The inequality $|t| < 5$ can be rewritten as "$t < 5$ and $t > -5$."

$$-5 < t \text{ and } t < 5$$

$$-5 < t < 5$$

As we learned in Section 2.3, this is a compound inequality, which uses the word "and." We must find a number that is greater than -5 but less than 5.

What if our inequality has a greater than sign? Let us try $|t| > 5$. This means we want the numbers greater than five spaces from 0 with respect to 0. That is all numbers greater than 5 and less than -5.



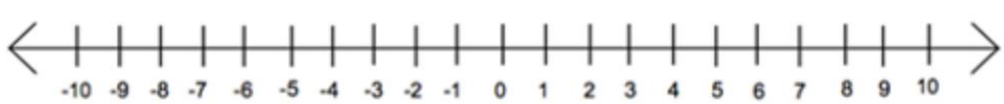
This time, we can think of the area between 5 and -5 as being fenced in. The goat can only go outside of it to the right, which would be greater than 5, or to the left, which would be less than -5. Written as an inequality, this would be $x < -5$ or $x > 5$. Again, as we learned in Section 2.3, this is a compound inequality, which uses the word "or." Our number will be in one or the other inequality but does not have to be in both at the same time. That would be the work "and".

Looking Ahead 2.5

Absolute value inequalities with the less than symbol use the word "and." Absolute value inequalities with the greater than symbol use the word "or."

Before we actually solve some problems, let us do a little more to understand what solving an inequality like $|t - 2| < 4$ means, or how it differs from an absolute value that has only a variable within the absolute value signs.

If the inequality given were $|t| < 4$, we would be looking for values whose distance is no more than 4 units to the left or 4 units to the right of 0: $-4 < t < 4$. We take the difference of two numbers to find the space between them. Because we have $|t - 2| < 4$, that means the values of t are numbers whose distance is 4 units to the left or right of 2. The numbers that work are between -2 and 6. Notice that it is between -2 *and* 6. The numbers -2 and 6 make the inequality false so it is the numbers between them, not including them, that make the inequality true.



Therefore, the solution is $-2 < t < 6$; or in terms of the original absolute value inequality, the solution is $-4 < t - 2 < 4$. This can be written as a compound inequality in order to solve it:

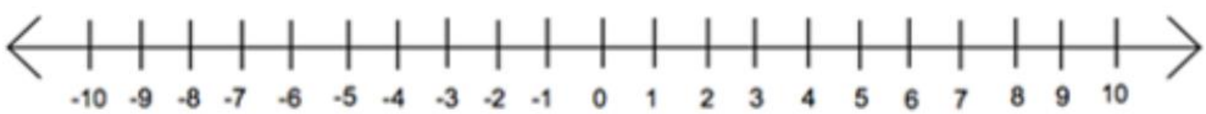
$$-4 < t - 2 \quad \text{and} \quad t - 2 < 4$$

$$+2 \quad\quad +2 \quad\quad\quad\quad +2 \quad +2$$

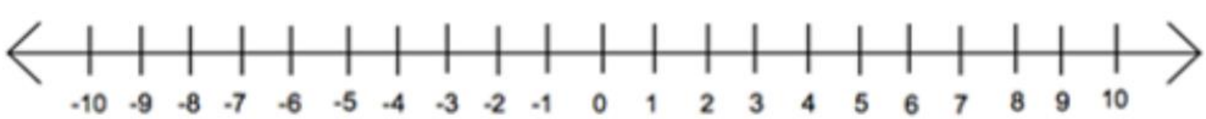
$$-2 < t \quad \text{and} \quad t < 6$$

$$-2 < t < 6$$

Example 1:	Solve $|t + 2| < 4$. To show all solutions, you must graph your solution.



Example 2:	Solve $|t - 2| > 4$ and graph your solution.



Example 3:	Solve for the variable t in $\|t+2\| \geq 4$ and graph your solution.

To summarize the rule for solving absolute value inequalities, let n be the variable expression.

If $|n| < b$ and $b > 0$
Then $-b < n < b$

If $|n| > b$ and $b > 0$
Then $n < -b$ or $n > b$

These are the same rules for $n \leq b$ or $n \geq b$, in which $b > 0$.
Note: b must be greater than 0 because absolute value never equals a negative number.

Section 2.6 Systems of Equations

Lesson Notes 2.6

We have solved equations and inequalities with one variable or more. We have explored equations and inequalities that contain absolute values. In *Pre-Algebra*, we solved equations with two variables, and graphed equations with two variables on the coordinate plane.

For the remainder of this module, we are going to find solutions for two equations or two inequalities that have two variables. This is called "Solving Systems of Equations/Inequalities" because there is more than one equation or inequality to solve.

There are several different methods we use to solve systems of equations/inequalities. We were introduced to these in *Pre-Algebra* and will investigate them deeper in the next few sections (2.7-2.10).

A good rule of thumb is to remember that if there is one variable, only one equation is needed to solve it; if there are two variables, two equations are needed to solve it; if there are three variables, three equations are needed to solve it; etc.

Looking Ahead 2.6

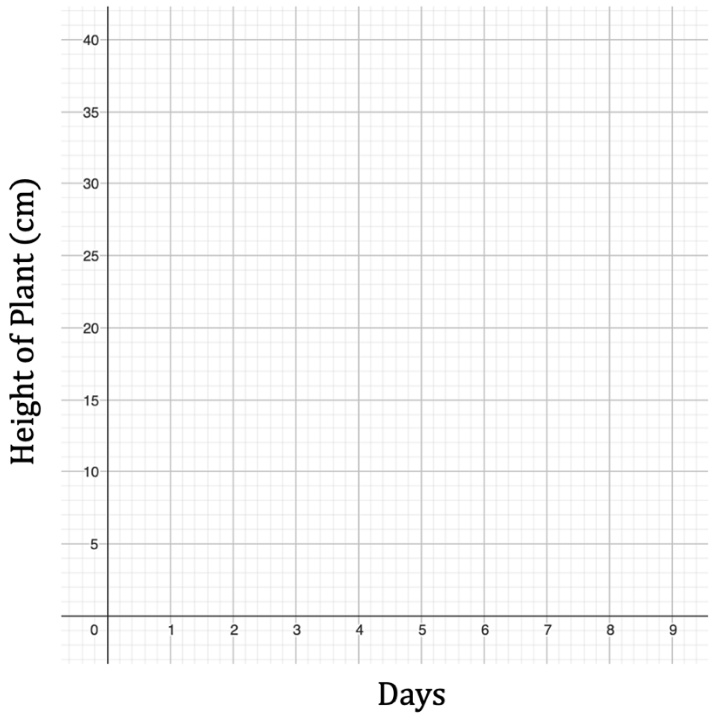
Two or more linear equations form a system of equations. A system of equations may have one solution, no solution, or infinite solutions.

The methods to solving a system can be graphing, substitution, or elimination. You will be able to determine which method works best for different systems of linear equations by the end of this module.

In *Algebra 2,* we will use matrices and substitution to solve systems involving three or more equations. For now, the remainder of this section will be devoted to looking into systems of equations.

Example 1: Hazel is doing a science fair experiment and testing plant fertilizer for plant growth. Plant Dynamo grows about 3 cm each day and is presently 5 cm. tall. Plant Exacto is growing about 4 cm each day and is presently only 1 cm tall. Hazel wants to determine what day they will be the same height and how tall they will be on that day. Write two equations for the growth of Plant Dynamo and Plant Exacto. After how many days will they be the same height and what will that height be?

Example 2: Complete the table and graph the equations for Plant Dynamo and Plant Exacto (from Example 1) and find the point of intersection. What is the significance of those coordinates?



	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Dynamo Height (cm)	5									
Exacto Height (cm)	1									

Section 2.7 Solving Systems of Equations Using a Graph

Looking Back 2.7

In the previous section, we investigated a real-world problem in which the solution was found by using a system of equations. When we graphed the growth for Plant Dynamo and Plant Exacto, their lines crossed at the point $(4, 17)$. This meant that by the fourth day of plant growth both plants were 17 centimeters tall. The point of intersection was the solution to the linear system of equations.

Systems of equations can also have no solution or infinite solutions. These will be investigated in this section using the graphing method. There are two other methods that will be explored in the following two sections: substitution and elimination.

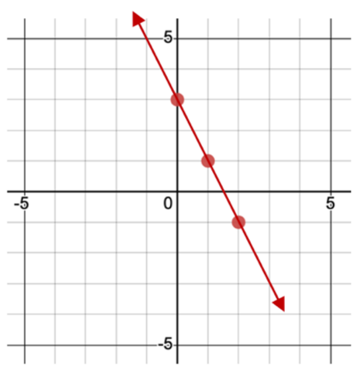
Let us solve the following system of equations using the graphing method:

$$2x + y = 3$$
$$y = -2x + 3$$

To graph the line $2x + y = 3$, create a table and substitute values in for x to find y, or substitute values in for y to find the x-values that make the equation true. For now, substitute in 0, 1, and 2 in for x and find y.

x	y
0	
1	
2	

The graph of the three points is the line below.



The graph is a line and there are an infinite number of sets of points on the line. Each of these points is a solution for the equation. We could substitute decimals, fractions, and/or negative numbers in for x and solve for y, but positive whole numbers are the easiest to work with and solve.

Now, graph the other equation in the system:

$$y = -2x + 3$$

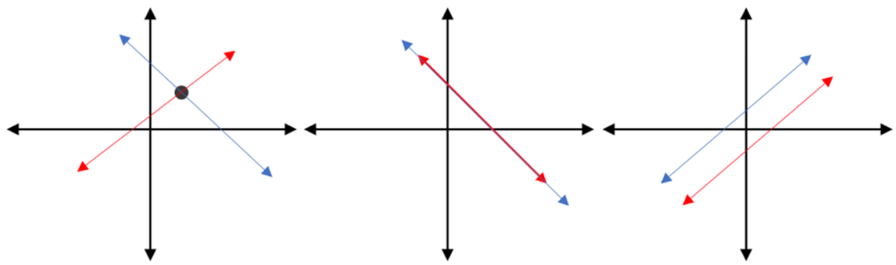
The first equation is in standard form but the second equation is in slope-intercept form with a slope of $m = -2$ and a y-intercept of $b = 3$. We can use these to plot two points on the previous graph using a different color and find where the lines intersect.

Both of the equations above are the same; one line lies on top of the other, which means the two equations have all points in common and an infinite set of solutions.

Looking Ahead 2.7

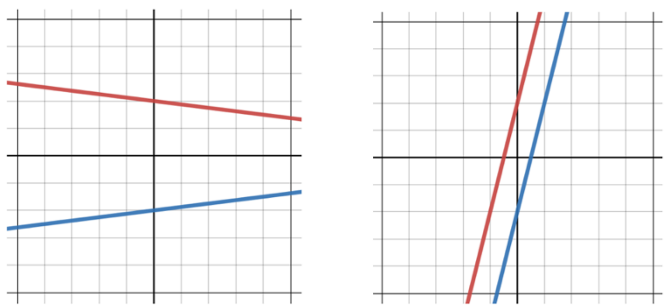
When we graph a system of equations, we have three possible solutions, no solutions (no points of intersection), infinite solutions (all the points on the line), or one solution (the point of intersection).

Example 1: How many solutions do each of the equations graphed below have?



___________ SOLUTION(S) ___________ SOLUTION(S) ___________ SOLUTION(S)

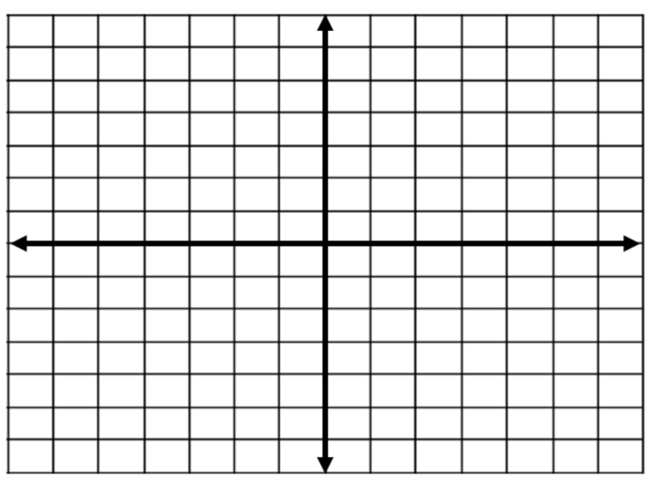
Example 2: How many solutions does each system of equations below have?



___________ SOLUTION(S) ___________ SOLUTION(S)

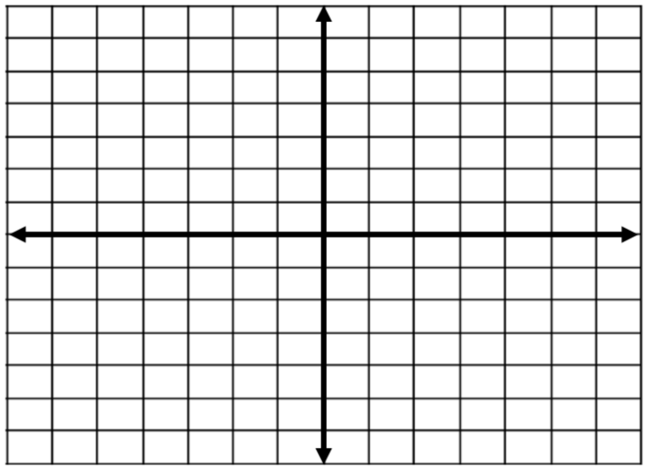
Example 3: Find the coordinates of the point that is a solution of the set of the two linear equations below.

$$y_1 = x + 1$$
$$y_2 = 2x - 3$$



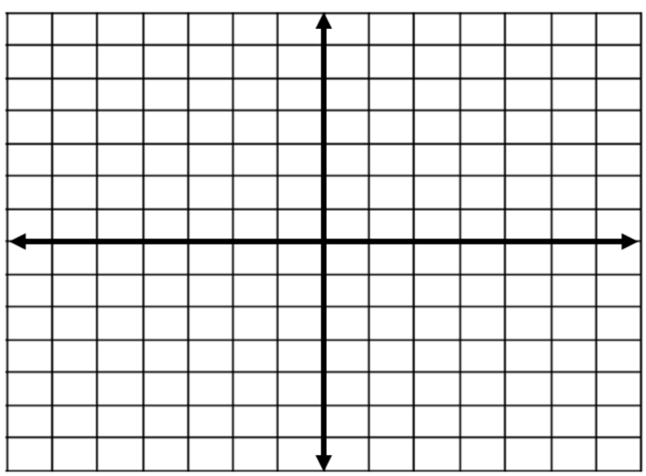
Example 4: Find the coordinates of the point that is a solution of the set of two linear equations below.

$$3x + 2y = 1$$
$$y = -3x - 1$$



Example 5: Find the coordinates of the point that is a solution of the set of two linear equations below.

$$3x - y = -5$$
$$y = 3x - 2$$

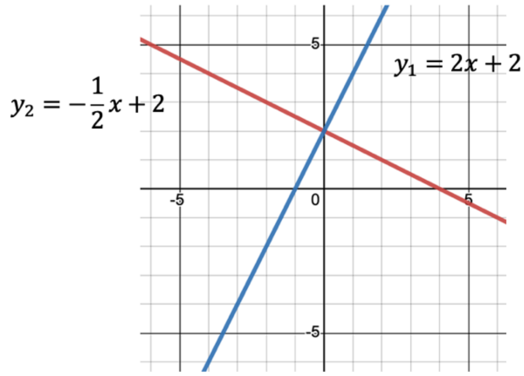


Section 2.8 One Solution, No Solution, or Infinite Solutions

Looking Back 2.8

In Problem 6 of the previous Practice Problems section, there was no point of intersection for the two parallel lines and therefore, no solution for the system of equations. This occurs when two or more lines have the same slope; if they have the same angle of inclination, they will rise or fall at the same rate. No matter how far apart they are or how close together they are, the slope remains constant.

Notice in the graph below the lines are perpendicular, which means they meet at a 90° angle (right angle).



Both lines have the same y-intercept of 2. The one called y_1 has a slope of 2 and the one called y_2 has a slope of $-\frac{1}{2}$. We call 2 and $-\frac{1}{2}$ opposite reciprocals. Now we have another rule:

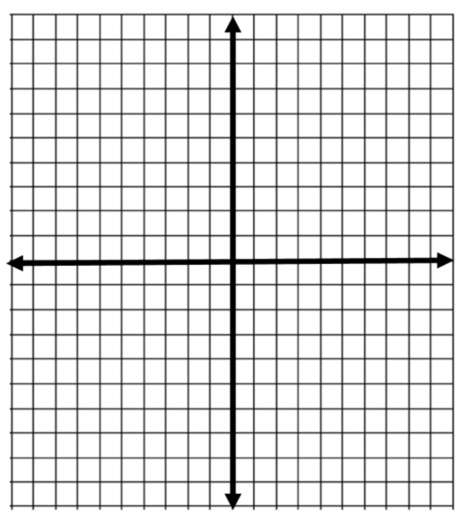
> **If equations have the same slope they are parallel.**
>
> **If equations have slopes such that one is the opposite reciprocal of the other, they are perpendicular.**

If you graph two equations and they are the same line, then one lies on top of the other. Because every point on the line is a solution for both equations, that means there are infinite solutions. Not only are the points on the lines solutions, but they extend infinitely in both directions. These lines would not only have the same slope, but also the same y-intercept. They are the same line.

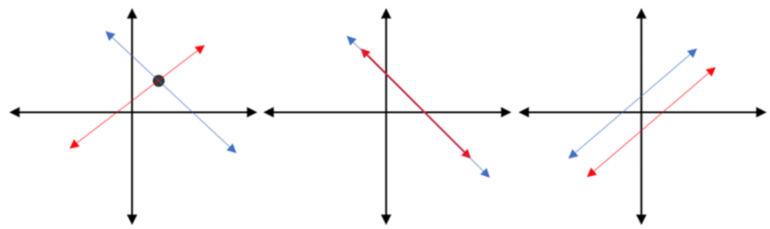
Example 1: Make a table for the standard form equation and plot the points, plot the y-intercept, and plot the slope to graph the second equation and demonstrate that both are the same equation and have infinite solutions.

$$\begin{cases} 2x + 4y = 12 \\ y = -\frac{1}{2}x + 3 \end{cases}$$

x	y
-4	
-2	
0	
2	
4	



Example 2:	Fill in the blanks to summarize the number of solutions for systems of equations.



Intersecting Lines	The Same Line	Parallel Lines
Slopes are ________ and the y-intercepts are the _______or________.	Slopes are the ______ and the y-intercepts are the _______.	Slopes are the _______ and the y-intercepts are ________________.
_______ Solution(s)	_______ Solution(s)	_______ Solution(s)

Section 2.9 Solving Systems of Equations Using the Substitution Method

Looking Back 2.9

Gabriel Cramer was a mathematician in Geneva in the 1700s who did much work with solving systems of equations. Cramer and another man named Calandrini shared the position of chairmen of the mathematics department at Académie de Calvin in Geneva at very young ages. They set up a schedule in which one would travel while the other would teach mathematics courses for a set time and then they would trade duties. This allowed Cramer to travel through Europe and meet many mathematicians including Euler, de Moivre, Stirling, and Daniel and Johann Bernoulli.

Throughout history, mathematicians and scientists have corresponded with one another to come up with multiple solutions to problems. Just as we have seen that systems of equations could be solved using graphs, they can also be solved by using a method called substitution. We have seen the method of substitution used when we set up equations equal to one another when they were both equal to y. Because y was in terms of x, the two equations only had one variable to solve for, x.

Looking Ahead 2.9

In Problem 4 of Practice Problems section 2.7, you drew graphs of the following two equations:

$$\begin{cases} y_1 = x + 1 \\ y_2 = 3x - 3 \end{cases}$$

You found the point of intersection to be (2,3). Therefore, (2,3) is a solution to the system of linear equations. The coordinates (2,3) are solutions to both equations. Each equation has infinite solutions, but there is only this one they share in common.

Another method to solve a system of equations is substitution. In using substitution, you set $y_1 = y_2$, then you can solve the equations.

Example 1:	Use the substitution method to solve the system of equations below.

$$\begin{cases} y_1 = x + 1 \\ y_2 = 3x - 3 \end{cases}$$

Did you get the same solution (using substitution) as you did when you graphed the system of equations? Check your solution.

Example 2: Use the substitution method to solve the system of equations below. (Substitute the value for y in the first equation and solve for x.)

$$\begin{cases} y = 3x - 4 \\ y = -5 \end{cases}$$

Example 3: Use the substitution method to solve the system of equations below. (Substitute the expression for y in the first equation and solve for x, then substitute x in either equation and solve for y; why is the second equation easier to use in this case?)

$$\begin{cases} 2x + 2y = 5 \\ y = -3x + 2 \end{cases}$$

Example 4: Use the substitution method to solve the system of equations below.

$$\begin{cases} 2x + 2y = 5 \\ 3x + y = 2 \end{cases}$$

Steps for Substitution:

Solve one equation for one variable in terms of the other variable.

Substitute that expression in for the variable in the other equation so there is only one variable in the equation now, not two.

Solve for the variable to find the actual value of the variable.

Substitute that value for the variable it represents into either of the other two equations to solve for the other variable.

Section 2.10 Solving Systems of Equations Using the Elimination Method

Looking Back 2.10

The Addition and Subtraction Property of Equality states that if two expressions are equal and two other expressions are equal, then the other expressions can both be added or subtracted to the original expressions and they remain equal.

If $a = b$ and $c = d$, then $a + c = b + d$

If $a = b$ and $c = d$, then $a - c = b - d$

With these properties, if there are two equations in a system and either the coefficients of the xs or the coefficients of the ys are opposites (for example: $6x$ and $-6x$ or $-3.2y$ and $3.2y$), they will cancel each other out when added together and only one variable will be left. Then we can solve for the remaining variable. Once the solution is found, the other variable can be found using substitution.

Looking Ahead 2.10

Example 1: In the system of equations below, eliminate one variable and use substitution to solve for the other variable.

$$\begin{cases} 3x - 11y = 28 \\ 4x + 11y = -14 \end{cases}$$

The Multiplication and Division Property of Equality states that if two expressions are equal and two other expressions are equal, then the other expressions can both be multiplied and divided to the original expressions and they remain equal.

If $a = b$ and $c = d$, then $a \cdot c = b \cdot d$

If $a = b$ and $c = d$, then $\frac{a}{c} = \frac{b}{d}$

We can multiply both sides of an equation by a number so that either the coefficients of x or y become the same or opposite of those in the second equation. Therefore, one variable is eliminated when the equations are added or subtracted.

Example 2: A local theatre charges $3 for tickets for children and $5 for tickets for adults. The theatre has 50 balcony seats and 82 lower-level seats. Suppose a matinee was sold out and the theatre made $484. How many tickets for adults were sold and how many tickets for children were sold?

Let a be the number of tickets for adults sold and let c be the number of tickets for children sold.

First, look at the number of seats in the theatre and write an equation to find the type of tickets that were sold.

Secondly, write an equation using the total amount of money made on ticket sales.

If both variables are eliminated and the combination of the two equations becomes $0 = -4$, then there is no solution because $0 \neq -4$.

If both variables and the numbers are eliminated and the combination of the two equations becomes $0 = 0$, then there are infinite solutions.

Because our variables are a and c, you might think you cannot write your solution as an ordered pair, but you can. The rule is to put the variables in the ordered pair in alphabetical order; therefore, we have (a, c).

Section 2.11 Graphing Linear Inequalities

Looking Back 2.11

We have solved and graphed equations, inequalities, and compound inequalities on a number line when they have one variable. We have graphed equations on the coordinate plane when they have two variables. We have graphed equations using tables, and using slope and y-intercept.

In this section, we will look at graphing inequalities on the coordinate plane. As with inequalities on the number line in which there are more than one solution, we must use shading to show all possible solutions.

Looking Back 2.11

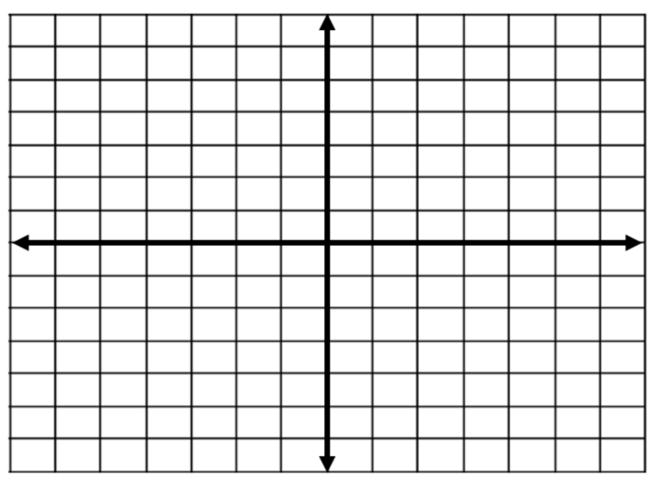
When $x > 3$, we should shade everything to the right of 3 on the number line. There is an open circle at $x = 3$ to show that it is not included. When $x \leq -1.5$, we shade everything to the left of -1.5 on the number line. There is a closed circle at $x = -1.5$ to show that it is included.

When we have an inequality with two variables such as $y \leq \frac{1}{4}x - 6.2$, that means we graph the line and shade everything below it. The corresponding line is called a "boundary line." For linear inequalities, the line is solid to show the solution does include the points on the line ($\leq$ or $\geq$) or the line is dashed to show the solution does not include the points on the line ($<$ or $>$).

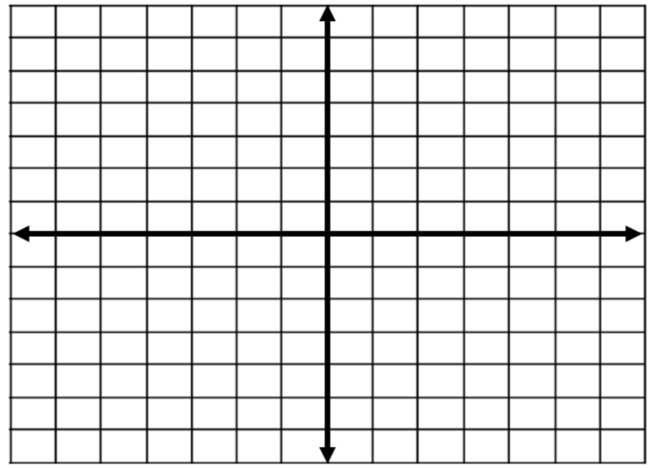
Follow the given steps to graph an inequality:

- Graph the boundary line that corresponds with the inequality.
- Decide whether the line is solid or dashed.
- Decide which side of the boundary line to shade.
  a) If the inequality is in slope-intercept form and the inequality is $y >$ or $y \geq$, then shade above the line.
  b) If the inequality is in slope-intercept form and the inequality is $y <$ or $\leq$, then shade below the line.
     a. Solve for one point above the line and one point below the line to check your shading. If the point above the line works, the shade should be above the line. If the point below the line works, the shade should be below the line.

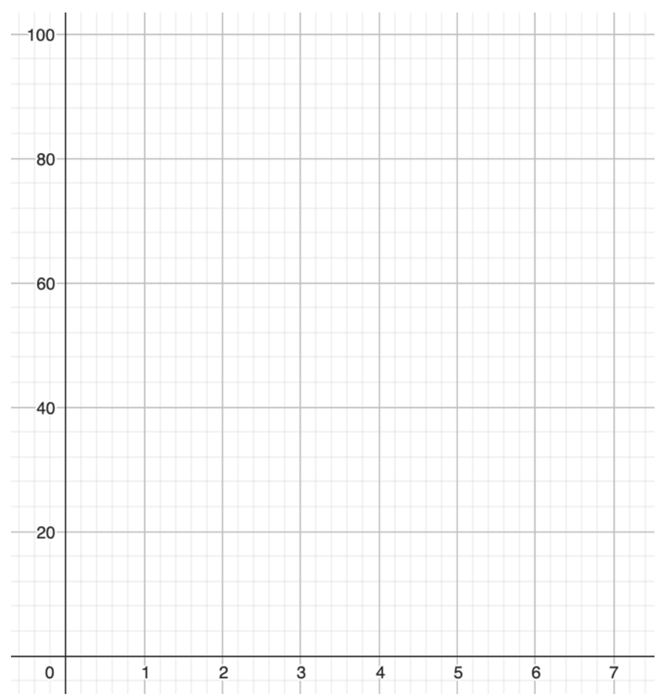
Example 1:	Solve $y > 2x + 3$ and graph your solution on the coordinate plane.



Example 2: Solve $2x - y \geq 4$ and graph your solution on the coordinate plane.



Example 3: Jacqueline gets a new job that pays commission. She gets a $50.00 sign on bonus and can make more than $10.00 per hour. Set up and solve the inequality and graph the solution. Let $y =$ money earned and $x =$ hours worked.

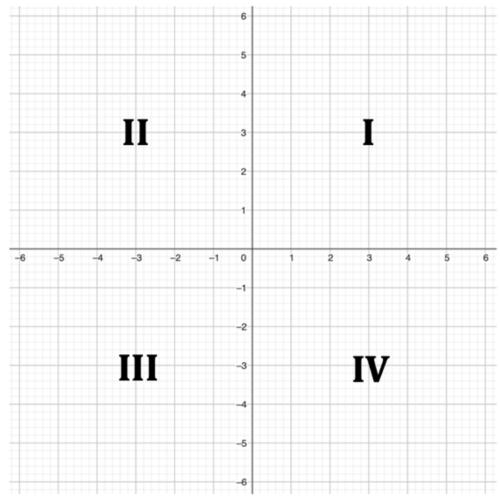


Section 2.12 Solving Systems of Inequalities

Looking Back 2.12

Previously, we played a game with a partner called "Where is Hinkle Hiding?" The idea of the game is for one partner to hide Hinkle on the coordinate grid and the other partner to find the location of Hinkle with the least number of guesses.

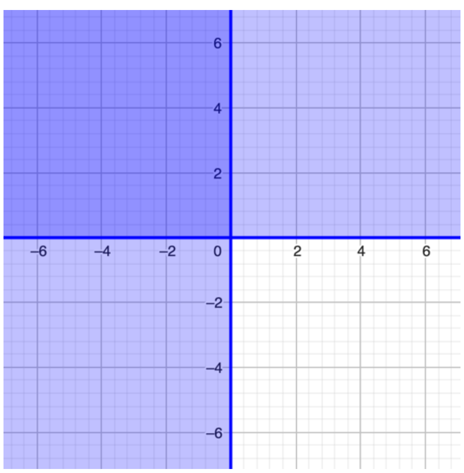
Trying to find Hinkle in any of the four quadrants in the coordinate grid makes things difficult. In the version of "Where is Hinkle Hiding?" explained below, only one question may be asked at a time and no hints are given, only "yes" and "no" answers. The goal is to find Hinkle in less than twenty questions. An example is given below.



The quadrants to the right are numbered I-IV, counter-clockwise.

Using the coordinate grid, we know Hinkle is hiding between -6 and 6 on the x-axis and -6 and 6 on the y-axis. The number -6 is considered the minimum for x and y on their respective axes. The number 6 is considered the maximum for x and y on their respective axes.

If you ask: "Is Hinkle at $x \geq 0$?" and the answer is "no," then Hinkle is in Quadrant II or III, but not on the y-axis.



The next question may be: "Is Hinkle at $y \geq 0$?" and if the answer is "yes," then you know Hinkle is hiding in Quadrant II or on the x-axis of Quadrant II.

If Hinkle is not at $x \geq 0$, then he will be at $x < 0$ or to the left of the y-axis. If Hinkle is at $y \geq 0$, and at $x < 0$ at the same time, then he is at the intersection of the quadrants or axes (where they overlap). The double-shaded area is the area in which he is hiding.

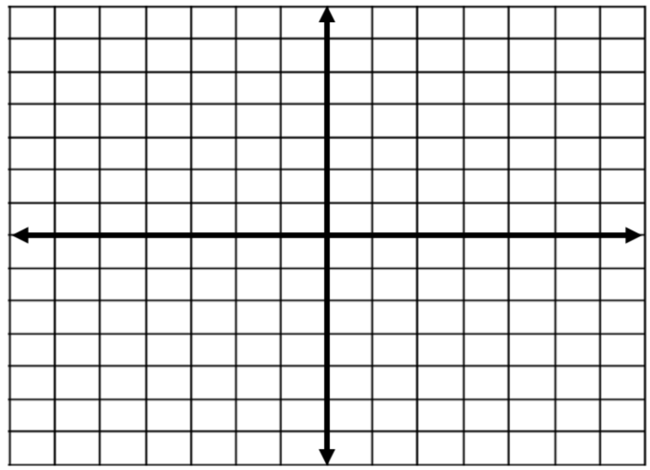
If you ask "Is Hinkle at $x \leq -3$?" and the answer is "yes," then Hinkle is in the left half of Quadrant II; if the answer is "no," then Hinkle is at $x > -3$, which is the right half of Quadrant II, not including -3. The question: "Is Hinkle hiding at $x > -3$?" or "Is Hinkle hiding at $x < -3$?" may be asked to see whether the axis is or is not included. When the plane in which Hinkle is located gets narrow, the specific coordinates can be asked of where Hinkle is located. You may want to try this with a partner now.

Looking Ahead 2.12

To find the intersection of $2x + y > 4$ and $x + 3y < 6$, both graphs must be drawn and shaded on the same coordinate grid. The boundary lines (corresponding equations) of this system of inequalities are dotted because they are not included in the solution. The overlap of the shaded regions of the two inequalities will be the solution to the inequalities. To graph inequalities using the slope and y-intercept, solve for y in terms of x first.

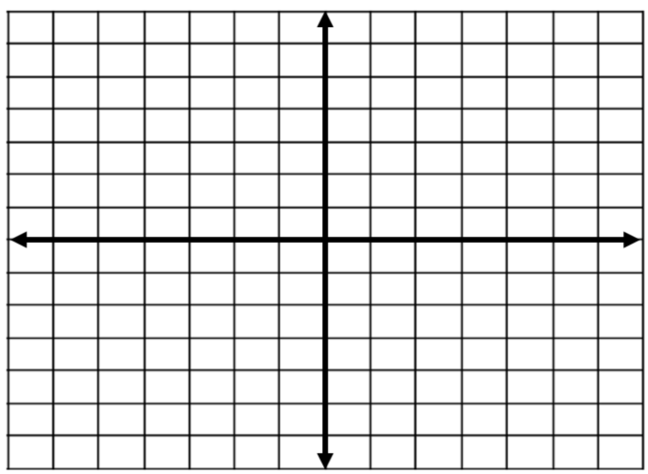
Example 1: Graph the solution set of the system of linear inequalities below.

$$\begin{cases} 2x + y > 4 \\ x + 3y < 6 \end{cases}$$



Example 2: Graph the solution set of the system of inequalities below.

$$\begin{cases} x \leq 0 \\ y \geq 0 \\ 3x + 4y \leq 12 \end{cases}$$



Section 2.13 Optimization Problems

Looking Back 2.13

Just as systems of linear equations can be used to solve real-world problems, systems of linear inequalities can be used to solve real-world problems, particularly problems of optimization. A mathematical optimization problem is a problem that considers all the possible alternatives or conditions (often called "constraints" or "parameters") of a real-world problem in order to find the best solution to the problem. This gives the optimal range for solutions when there are several possible solutions. The graph of the solution is called the feasible region, which represents all possible solutions to the inequalities, or those solutions (regions) that are feasible.

Looking Ahead 2.13

The equation of an optimization problem is called the linear program. The intersection points are called the corner points and represent the minimum and maximum points in the optimal region (solution set).

Example 1:	In the system of inequalities below, graph the feasible regions and identify the corner points.

$$x \geq 2$$
$$x \leq 4$$
$$y \geq -2$$
$$y \leq -x + 2$$

There are three corner points where the lines intersect:

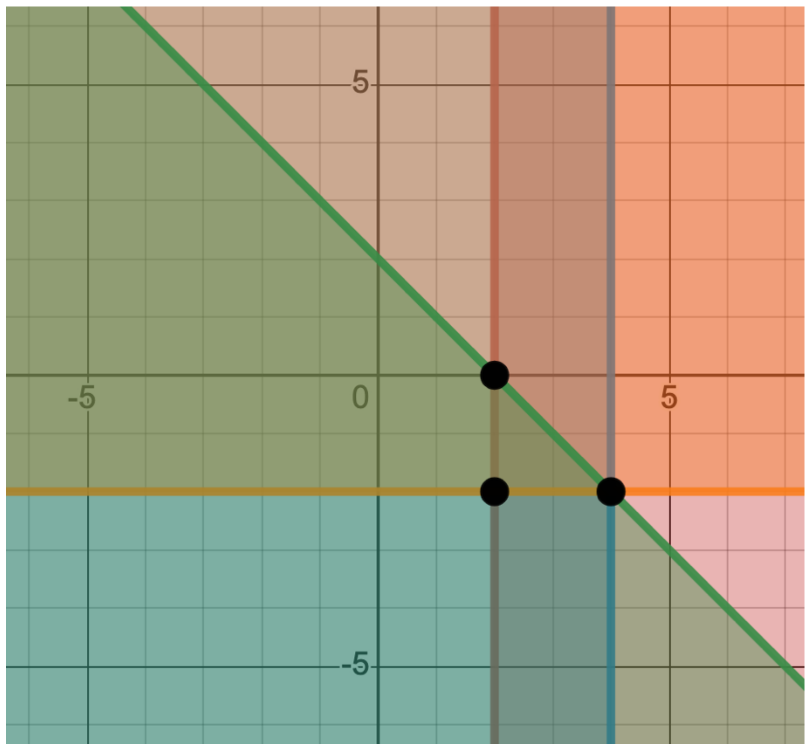
____

____

____

All of these satisfy the above

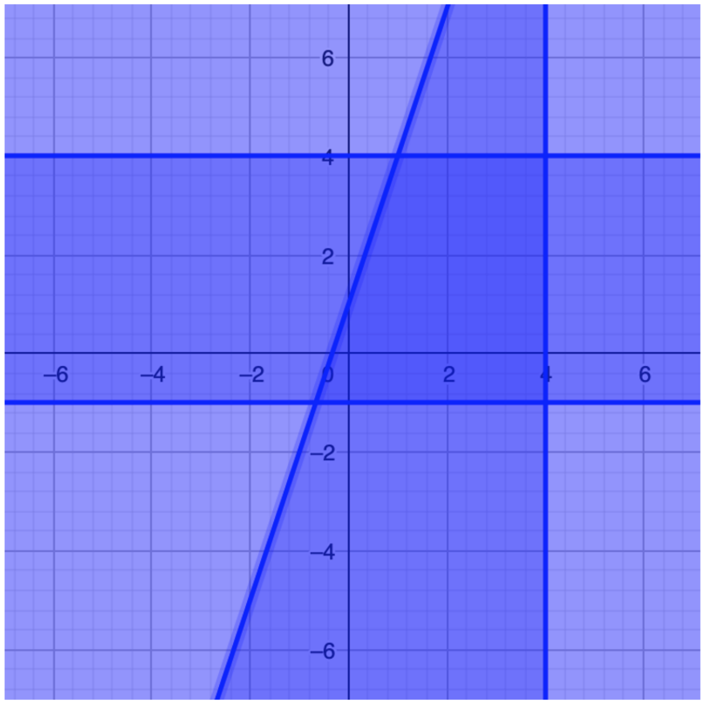
________________________________.



Example 2:	Describe the constraints on the feasible region for the graph below. Locate the corner points.

b) $x \leq 4$

a) $y \leq 4$



c) $y \geq -1$

d) $y \leq 3x + 1$

The equations are the ______________________________.

The corner point of lines a) and b) is ____________________.

The corner point of lines b) and c) is ____________________.

The corner point of lines a) and d) is ____________________.

The corner point of lines c) and d) is ____________________.

Example 3: A parking garage has an area of 640 square meters. A minivan requires 8 square meters of space and a truck requires 32 square meters of space. The garage can park a maximum of 50 vehicles. If v represents minivans and t represents trucks, the inequalities below represent the parking garage problem.

$$v + t \leq 50$$
$$8v + 32t \leq 640$$

If a minivan costs $\$5.00$ to park and a truck costs $\$10.00$ to park in the garage, what is the maximum number of minivans and trucks that can be parked for the garage to make a maximum amount of profit? What is the maximum amount of profit?