Geometry and Trigonometry Module 7 The Unit Circle and Trigonometric Functions

Section 7.1 Trigonometric Ratios for Right Triangles

Looking Back 7.1

How did the Egyptians build the pyramid with no protractors? How did all angles in the pyramids remain the same without the use of these tools?

As you learned in the previous module, the Egyptians made a knot in a rope and folded the length from the beginning of the rope to the knot over onto itself so the units of these lengths would be the same. The knotted rope of 3-4-5 (3 knots, 4 knots, 5 knots) would form a 90° angle at the center. The Egyptians were probably aware of the Pythagorean Theorem and of Pythagorean Triples.

It should be clear by now that advanced geometry and trigonometry are very evident in the pyramids.

A knotted rope with a 90° angle at the center and legs of the same length, say 2-2 or 3-3, etc., forms an isosceles right triangle whose ratio is the tangent function and with base angles of 90°:

$$\tan 45° = \frac{1}{1} \qquad \tan 45° = \frac{2}{2} \qquad \tan 45° = \frac{3}{3}$$

The Egyptians discovered many angles associated with ratios and recorded them in tables on stones or clay. Years later, but before the use of calculators, these tables were often found at the back of trigonometry books. Now, they are stored in graphing calculators.

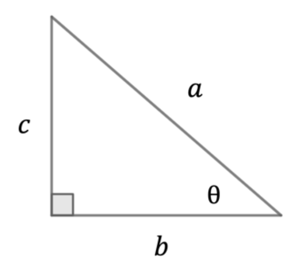
Looking Ahead 7.1

You discovered the trigonometric ratios for right triangles in the previous module. These are shown as follows:

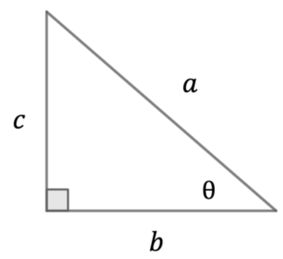
$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

The acronym used to remember this is as follows:

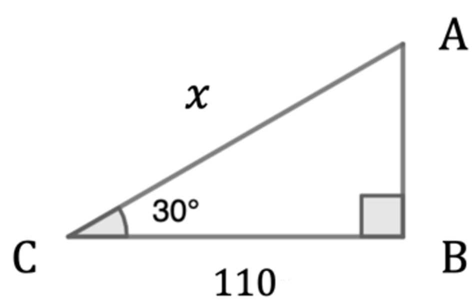
Soh-Cah-Toa



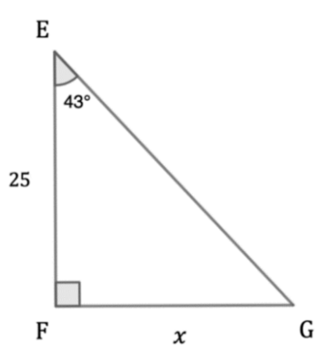
Example 1: Find the trigonometric ratios in the given triangle.



Example 2: Solve for x in the given triangle.



Example 3: Solve for x in the given triangle.



Section 7.2 Inverse Trigonometric Functions

Looking Back 7.2

Inverses have been used in algebra to solve for unknown variables. The inverse of addition is subtraction; the inverse of multiplication is division. Powers and roots are inverses of one another. The inverse of a square is a square root; the inverse of a cube is a cube root.

Exponential functions and logarithmic functions are used to solve for one another. The trigonometric functions of sine θ, cosine θ, and tangent θ and the inverse trigonometric functions are also used to solve for one another. These inverse trigonometric functions are represented by special characters on the calculator:

- The inverse of sine is sin^{-1}
- The inverse of cosine is cos^{-1}
- The inverse of tangent is tan^{-1}

This can be confusing because $x^{-1} = \frac{1}{x}$; however, $\sin^{-1}(x) \neq \frac{1}{\sin x}$. These are not inverses. This is not the meaning of the notation used.

Because of this confusion, the inverse of sine is sometimes called arcsine, while the inverse of cosine is arccosine and the inverse of tangent is arctangent.

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \text{so, } \theta = \sin^{-1}(\frac{\text{opp.}}{\text{hyp.}})$$

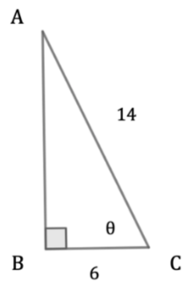
$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \text{so, } \theta = \cos^{-1}(\frac{\text{adj.}}{\text{hyp.}})$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} \qquad \text{so, } \theta = \tan^{-1}(\frac{\text{opp.}}{\text{adj.}})$$

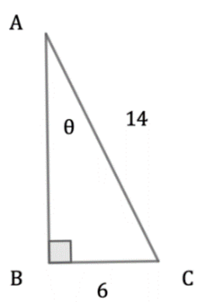
In problems in the previous section, the angle was given in right triangles and the trigonometric functions were used to find the missing sides. In problems in this section, sides are given in right triangles, and the inverse trigonometric functions are used to find the missing angles.

Looking Ahead 7.2

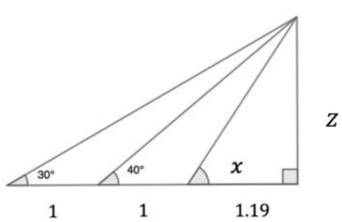
Example 1:	Find the missing angle in the given triangle.



Example 2:	Find the missing angle in the given triangle.



Example 3:	Solve for x and z in the figure below. Round to the nearest tenth.



Section 7.3 Special Right Triangle Ratios

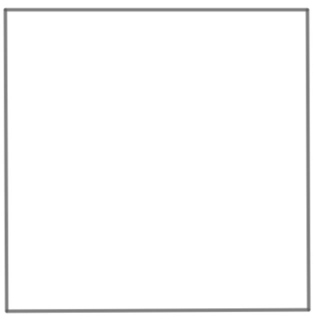
Looking Back 7.3

There are two triangle ratios often called special triangle ratios because their ratios are easy to calculate and they are seen repeatedly in the unit circle, which will be introduced in the next section. These special triangles are the $30° - 60° - 90°$ right triangle and the $45° - 45° - 90°$ triangle, which is an isosceles right triangle.

You have been introduced to these triangles in practice problems from the previous module. Here, we will investigate where the ratios come from and how to use them.

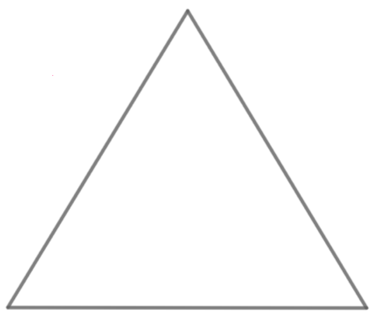
Looking Ahead 7.3

Example 1: Below is a square with a side length of 1 unit. Use a ruler to draw one diagonal in the square. This will form two congruent right angles. Label the angle measures in one of the triangles. Find and label the lengths of all three sides of one of the triangles. Use this information to find the sine, cosine, and tangent ratios for the non-right angles.

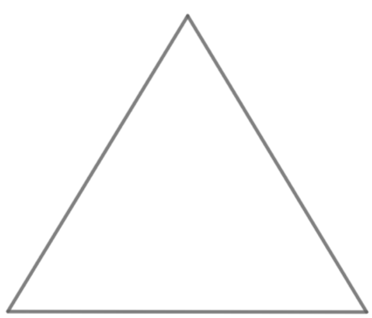


Example 2: Below is an equilateral triangle with side lengths of 2 units. Draw a perpendicular line from one vertex to the opposite side. What important things does this perpendicular line do to the equilateral triangle (in terms of angle measures and side lengths)?

Label each angle measure in one of the new triangles created by the perpendicular line. Find and label the length of each side of the new triangle. Use this information to find the sine, cosine, and tangent ratios for the non-right angles.



Example 3: What would happen to the trigonometric ratios in each new triangle (created in Example 2) if the original length of each side were changed? Let the side length be 3 units. What is the length of each of the legs?



Section 7.4 The Unit Circle and the Wrapping Function

Looking Back 7.4

A linear scale is used to measure time. Time is represented on the x-axis of the x-y coordinate plane. It is also represented by the circular motion of the second hand on a watch or clock.

In today's investigation of the unit circle, the wrapping function will be used to merge the linear scale with the circular motion.

Looking Ahead 7.4

1. Using a tin can, trace the circumference of the bottom on a piece of paper. Any size tin can will work.

2. Mark the center of the circle and draw the x-axis and y-axis through it. Label the center of the circle $(0, 0)$ for the origin.

3. Label the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$ on the circle. The point $(1, 0)$ represents an x distance of 1 unit, which is 1 radius of the circle.

4. Put a ruler along the right side of an 8.5" x 11" piece of white paper. Trace along the left side of the ruler and cut this piece out. If the can is small, 1 strip of paper will work. If the can is large, cut out another strip of paper the width of the ruler and tape the ends together.

5. Write in a "0" at the start of the strip and write in a "1" on the strip of paper the length of 1 radius of the circle. Fold this length over onto itself and label the folds "2," "3," "4," etc. for the length of each radius.

6. Find and mark $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, and 2π on the strip of paper; $\pi \approx 3.14$, so $\frac{\pi}{4} \approx \frac{3.14}{4} \approx 0.79$. This is approximately 0.8 units and just before the 1-unit mark on the strip of paper. These will be approximations and locations will be estimations.

7. Tape the strip of paper to the bottom of the tin can so the numbers are showing on the outside. Line up the 0 on the strip with point $(1, 0)$ on the coordinate grid. Center the can over the traced circle. Transfer the numbers on the strip of paper to the circle. The "1" now represents 1 radian on the unit circle; it is a radius of 1 on the coordinate grid. When finished, connect $(0, 0)$ to 1 radian. This is the central angle (in radians). Label it "θ." The central angle is measured counterclockwise from the positive x-axis. It is the ratio of the length of the intercepted arc to the radius of the circle. Notice that approximately 3 radians (precisely 3.14 or π radians) is about halfway around the circle at point $(-1, 0)$. So, π radians $= 180°$ and these are approximately equal to the decimal number 3.14. These three representations on the unit circle are in radians, degrees, and x-y coordinates. What is the angular and decimal representation of 2π radians?

8. Find and label these degrees (which are central angles) and mark them on the arc of the circle: $30°$, $45°$, $60°$, $90°$, $135°$, $180°$, $225°$, $270°$, and $360°$. Remember, the central angle is equal to the subtended arc and is measured counterclockwise from the positive x-axis.

9. Find and mark these radians on the circle (remember, 1 radian on the arc of the circle is the same as 1 radius of the circle): π, $(\frac{\pi}{2})$, $(\frac{\pi}{3})$, $(\frac{\pi}{4})$, $(\frac{3\pi}{2})$, 2π. Since you know $180°$ is the same as π radians, then you know $\frac{\pi}{2}$ is half of π, so if π is half-way around the circle, then $\frac{\pi}{2}$ is half of that ($\frac{1}{4}$ of the way around the circle), which is at the same place as $(0, 1)$. Again, there are three ways to name a point on the unit circle: exact points (or decimal approximations) on the x-y axis, degrees, and radians.

10. Let us see if you can remember the exact location of $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. It is the length of x and the height of y for the right triangle that has a central angle of 45°. Because both coordinates are positive, it is in Quadrant I of the unit circle. Find the ordered pair on the unit circle and label it. What about $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$? The former has a central angle of 30° and the latter has a central angle of 60° respectively. Find and label these points on the unit circle. Use the special right triangle ratios with the hypotenuse as 1 unit since it is the radius of the unit circle. Complete these in all four quadrants. Do you notice a pattern?

11. In each of the four quadrants, put what the signs of sine and cosine would be in terms either positive or negative. Remember, $\cos\theta = x$ and $\sin\theta = y$. Think what the x and y would be in each quadrant. So, for Quadrant IV, you would have $(+, -)$, which means cosine is positive and sine is negative.

12. Put the sign for the tangent function in each quadrant. Since $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$, it is the same as $\frac{y}{x}$ or $\frac{\sin\theta}{\cos\theta}$. So, for Quadrant IV, you are dividing a negative by a positive $(\frac{-}{+})$ and that gives you a negative. Therefore, in Quadrant IV, you would put "$\tan\theta$ is $-$ (negative)."

Section 7.5 The Unit Circle and the Smartie® Simulation

Looking Back 7.5

In the last section, tin cans and the wrapping function introduced us to the unit circle. In this section, paper, a compass, and Smarties® will be used to strengthen our understanding of the unit circle.

Follow the instructions to complete the project and answer the given questions.

1. Draw an x-y axis with origin (0, 0) on a separate sheet of paper.
2. Place 5 Smarties® along the positive x-axis (so that half of each Smartie® is on either side of the line). Put a mark at the end of the 5th Smartie® on the x-axis. Open a compass from the origin to the mark you just made and draw a circle with a radius of 5 Smarties®.
3. Beginning above the positive x-axis, where it crosses the circle, place 5 Smarties® along the arc in a counterclockwise direction (so that each half of the Smartie® is on either side of the line).
4. Draw a straight line from the origin to the end of the 5th Smartie®. The central angle of rotation from the positive x-axis to the line you have just drawn has an arc equal to 1 radius of the circle. This arc length is 1 radian.
5. How many Smarties® are equal to 2 radians? Add 5 more Smarties® to the arc of the circle, moving counterclockwise. Draw a line from the origin to the end of the 10th Smartie®. This is 2 radians.

6. How many Smarties® go halfway around the circle? About how many radii of the circle is that? About how many radians is that? How many degrees around the circle is this?

7. How many Smarties® go all the way around the circle? About how many radii of the circle is that? About how many radians is that? How many degrees around the circle is this?

8. Now, add 2 more Smarties® to the outside of the 5-Smartie® radius circle along the positive x-axis. The radius is now 7 Smarties®. Make a new mark on the x-axis at the end of the 7th Smartie®. Open a compass from the origin to the new mark and draw a 7-Smartie® radius circle so that it is concentric to the 5-Smartie® radius circle.
9. Beginning above the positive x-axis where it crosses the circle, place 7 Smarties® along the arc in a counterclockwise direction (so that each half of the Smartie® is on either side of the line).
10. Draw a straight line from the origin to the end of the 7th Smartie®. The central angle of rotation from the positive x-axis to the line you have just drawn has an arc equal to 1 radius of the circle. This arc length is 1 radian. What do you notice about the central angle for 1 radian in both the 5-Smartie® and 7-Smartie® circle?

11. How many Smarties® are equal to 2 radians? Add 7 more Smarties® to the arc of the circle, moving counterclockwise. Draw a line from the origin to the end of the 14^{th} Smartie®. This is 2 radians.

12. How many Smarties® go halfway around the circle? About how many radii of the circle is that? About how many radians is that? How many degrees around the circle is this?

13. How many Smarties® go all the way around the circle? About how many radii of the circle is that? About how many radians is that? How many degrees around the circle is this?

Looking Ahead 7.5

Mark radians 1-6 on the concentric circles you have drawn. About how many radian angles add up to one complete circle? What is the constant in the equation for the circumference of a circle?

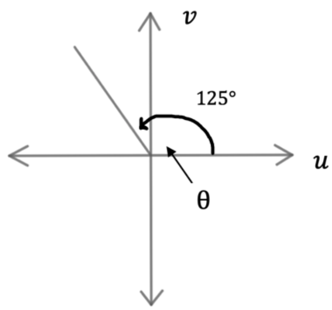
Mark the degrees and radians on the concentric circles at one-quarter, one-half, and three-quarters, and at the whole circles at one-quarter, one-half, and three-quarters.

Bisect the four quadrants of the circles and label the degree and radian measure of each central angle moving counterclockwise from the positive x-axis.

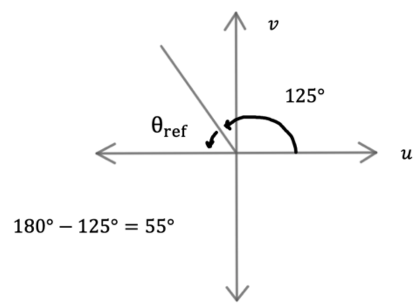
Section 7.6 Reference Angles, Radians, and Degrees

Looking Back 7.6

Sometimes, the x and y axes are labeled u and v, particularly in Pre-Calculus and Calculus when vectors are introduced or when investigating angle rotations. An angle θ is in standard position if the initial side (ray) is on the positive horizontal axis (u) and the vertex is centered at the origin. Angle θ is $125°$ and is standard position. An angle is quadrantal if its terminal side coincides with a coordinate axes. The angle is the measure of rotation between the two rays (initial side and terminal side) and the direction determines if the angle is positive or negative. A counterclockwise rotation results in a positive angle and a clockwise rotation results in a negative angle. Two angles are coterminal if they share a terminal side when in standard position.



The reference angle is the angle between the terminal side and the nearest side of u (the horizontal axis). This is labeled θ_{ref}. The reference angle $\theta_{\text{ref}} \leq 90°$ and is therefore positive. In Quadrant I, the reference angle is the same as the given angle.



You know that π radians is $180°$ and 2π radians is $360°$. These ratios are equivalent and form the proportion $\frac{\pi}{180} = \frac{2\pi}{360}$. Therefore, to convert from one to the other, set up proportions using these ratios and solve for the missing degrees or radians.

For example, to convert $125°$ to radians:

$$\frac{125°}{r} = \frac{180°}{\pi}$$

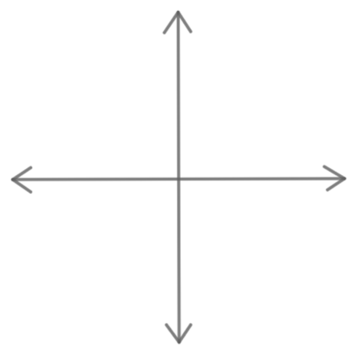
$$125\pi = 180r$$

$$\frac{125\pi}{180} = r$$

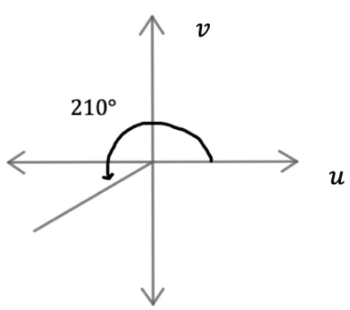
$$r = \frac{25\pi}{36}$$

Looking Ahead 7.6

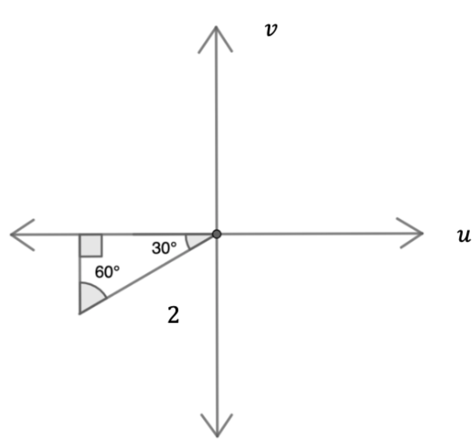
Example 1: Sketch an angle of $305°$ degrees in standard position. Sketch the reference angle and find its measure. Find the radian measure of the standard angle and reference angle.



Example 2: What is another name for an angle of $210°$ in standard position. Draw the reference angle (θ_{ref}) for the $210°$ angle. Give the radian measure for the standard angle and reference angle. Name a coterminal angle.



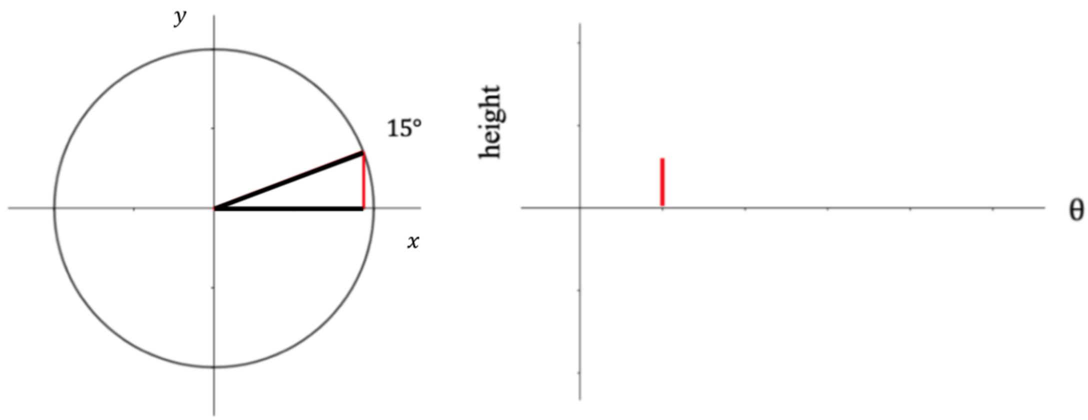
Example 3: The segment at $210°$ has a measure of 2 units. If a line is drawn from this segment perpendicular to the u-axis, what is the measure of the legs of the right triangle?



Section 7.7 The Sine and Cosine Functions

Looking Back 7.7

Use the unit circle and coordinate grid below to simulate the graph of the sine curve.



1. Use a protractor to mark every angle on the unit circle in $15°$ increments. Draw the radius from the origin to the point where the radius intercepts the arc of the circle. Then draw a line from this point of interception perpendicular to the x-axis. The first one is done for you at $15°$. This a triangle for an angle rotation of $15°$ in standard position.

2. Label the x-axis on the coordinate grid from $0°$ to $360°$ in intervals of $15°$ to represent these angles of rotation in the unit circle (θ). Use a toothpick to measure the radius of the unit circle (in 'toothpicks'). Transfer this height to the y-axis of the coordinate grid, marking a height of 1 radius above the y-axis and -1 radius below the y-axis.

3. Remember, since the sine function is the ratio of the opposite side over the hypotenuse of the triangle at the $15°$ angle, this height (shown above in red) would be the height the perpendicular segment ($y_{15°}$) over the radius length that is $\sin 15° = \frac{y_{15°}}{1}$ or $\sin 15° = \ y_{15°}$. Using the toothpick, measure the height of $y_{15°}$ and transfer it to the coordinate grid at the $15°$ mark. This is the sine ratio height at the $15°$ angle and is shown in red on the unit circle and graph.

4. Draw a second triangle at $30°$ and, using the toothpick, measure and transfer the $y_{30°}$ height (the side opposite the $30°$ angle) to the coordinate grid. Do this for all angles in Quadrants I and II. All these heights will be above the x-axis since the vertical displacement (height of y_θ) is positive.

5. Draw all triangles for Quadrants III and IV and transfer the sine height (y_θ) from the unit circle to the coordinate grid as before. Notice that the toothpick heights for y_θ will be below the y-axis since the displacement for these two quadrants is negative.

6. Place a point on the coordinate grid at $(0°, 0)$ and $(360°, 0)$ since the height of $y_{0°}$ and $y_{360°}$ is 0. There is no triangle at the start, end, or halfway around the circle. Half a revolution is $180°$ and a full revolution is $360°$. Beginning at $(0°, 0)$, draw a smooth curve connecting the ends of the positive and negative heights of all the lines for one full sine curve ending at $(0°, 360°)$.

Looking Ahead 7.7

Answer the following questions regarding the sine function simulation.

a) What do x and y represent on the curve? Fill in (x, y) in terms of the coordinate grid.

b) What is the equation that models the curve of the graph?

c) What is the maximum height of the graph? At what point does this occur? What is the minimum height of the graph? At what point does this occur?

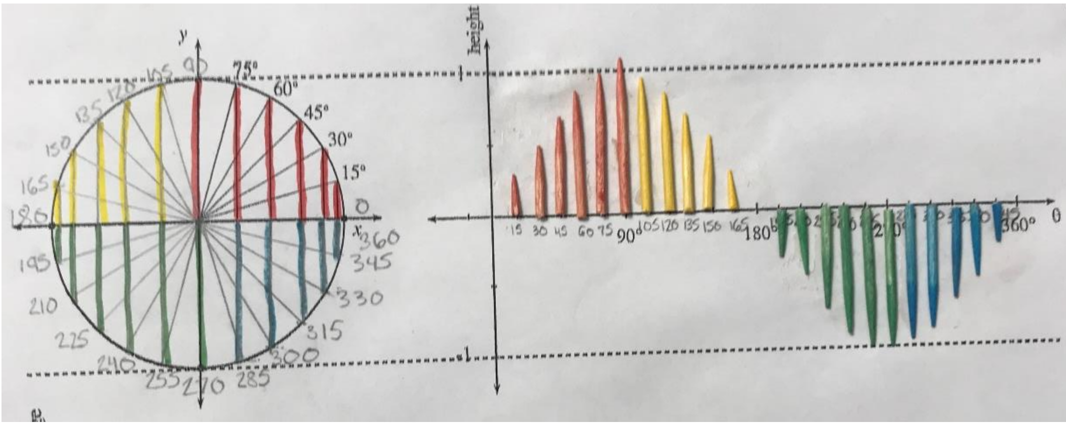
d) The mid-line is the horizontal axis that is halfway between the maximum and minimum values of the function. What is the equation for the mid-line?

e) The amplitude in a periodic function is half the distance between the highest and lowest point on the graph. This is also the distance from the mid-line to the maximum or minimum point. What is the amplitude of the function?

f) What is the domain of the sine function? What is the range of the sine function?

g) Write the interval for this graph. Would the graph change for angles over the interval $[360°, 720°]$?

Shown with toothpicks below is the completed graph of $y = \sin\theta$.



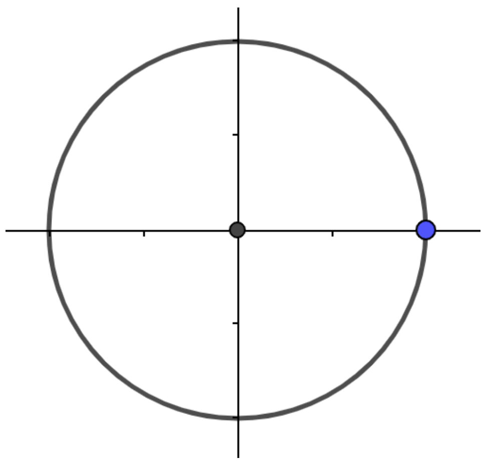
The cosine curve will be investigated in the Practice Problems section.

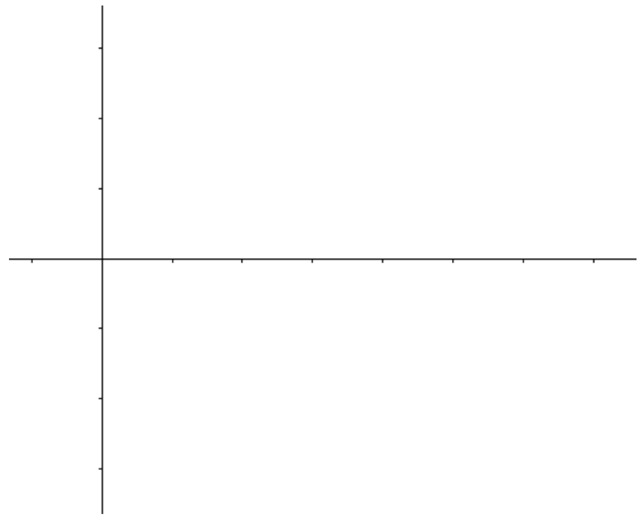
Section 7.8 The Tangent Function and Inverse Graphs

Looking Back 7.8

In this section, we will graph the tangent function using radians for the angle measure rather than degrees. There are twelve 15° angles formed from 0° to 180°; a 15° angle is one-twelfth of 180°, or $\frac{\pi}{12}$.

On the unit circle below, mark twelve angles in $\frac{\pi}{12}$ increments; and transfer the same increments to the tick marks on the coordinate grid beside the unit circle for Quadrant I.





1) Mark the radians of the central angles in $\frac{\pi}{12}$ increments for Quadrants II-IV. Draw a tangent line to the unit circle through point $(1, 0)$. Extend the radii of each triangle in Quadrant I so they intersect the tangent line.

2) Remember, the tangent ratio is the length of the opposite side to the length of the adjacent side of the central angle in the triangles. Since the adjacent side stays 1 unit in length, the tangent is the height of the side opposite the central angle.

3) Transfer each point of intersection to the same height on the coordinate graph above the corresponding central angle in standard position.

4) Connect the dots from $(0, 0)$ to form a smooth curve; this is the tangent function for Quadrant I.

5) The tangent values are negative in Quadrant II. Draw five triangles in Quadrant II congruent to those in Quadrant I for angles $\frac{7\pi}{12}$ to π every $\frac{\pi}{12}$ interval. Extend the radii through Quadrant III and onto the tangent line, which is parallel to the y-axis. These points are below the x-axis. Transfer them below the angles they correspond with, which are found on the x-axis of the coordinate grid.

6) Extend the triangle lines from Quadrant I to Quadrant III. Tangent is also positive in Quadrant III. Repeat the output values for $\frac{13\pi}{12}$ to $\frac{3\pi}{2}$ for those from 0 to $\frac{\pi}{2}$. The lines at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are parallel to the tangent line and do not intersect it because tangent is undefined at points $(0, 1)$ and $(0, -1)$. This is because tangent $\frac{\pi}{2} = \frac{1}{0}$ and tangent $\frac{\pi}{2} = \frac{-1}{0}$, which are both undefined. Put dashed lines through $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ to show there are asymptotes there. Connect the transferred points to make a smooth curve.

7) Quadrant IV is a repeat of the output values for Quadrant II and the points are found below the x-axis.

8) There are two x-intercepts on the unit circle at $(1, 0)$ and $(-1, 0)$. These are at 0, π, and 2π radians. Tangent $\pi = \frac{0}{-1}$, which is 0, and tangent $2\pi = \frac{0}{1}$, which is also 0. Mark these x-intercepts at $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$ on the coordinate grid.

Looking Ahead 7.8

In God's universe, many patterns may be found in periodic functions and phenomena that are cyclic.

Let us just look at the values found on the graphing calculator, which are stored for these trigonometric keys. Make sure your calculator is in 'radian' mode.

Quadrant I	
θ	**tan θ**
$\frac{\pi}{6}$	0.577
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	1.73

Quadrant II	
θ	**tan θ**
$\frac{2\pi}{3}$	-1.73
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	-0.577

Quadrant III	
θ	**tan θ**
$\frac{7\pi}{6}$	0.577
$\frac{5\pi}{4}$	1
$\frac{4\pi}{3}$	1.73

Quadrant IV	
θ	**tan θ**
$\frac{5\pi}{3}$	-1.73
$\frac{7\pi}{4}$	-1
$\frac{11\pi}{6}$	-0.577

Observations of the tangent function:

1) $\tan(-x) = -\tan x$ because the graph of $y = \tan x$ is symmetric about the origin

$$\tan x = \frac{\sin x}{\cos x}$$

2) $\tan x = 0$ when $\sin x = 0$ at $x = \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ etc.

3) the domain is all real numbers except when $x = \frac{\pi}{2} + \pi n$, where n is an integer

4) $\tan x$ is undefined when $\cos x = 0$ at $x = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ etc. … these are vertical asymptotes

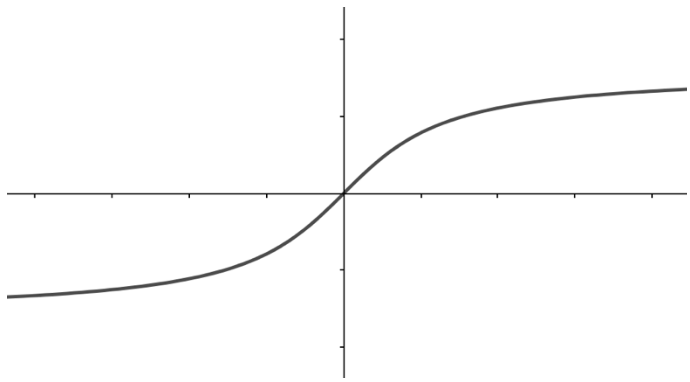
5) $\tan(x + \pi) = \tan x$ when $\tan x$ is not undefined because the period of $\tan x$ is π

6) the range of $\tan x$ is all real numbers since its end behavior is unbounded.

To find the inverse graph of the tangent, switch the θ and tan θ values in the table and plot the points.
The range of any trigonometric function becomes the domain of its inverse.
Range (y) = Domain (y^{-1})
So, the domain for the tangent function is as follows:

Function	**Domain**
$y = \tan^{-1}(x)$	$-\infty < x < \infty$
$f(\theta) = \tan^{-1}(\theta)$	all real numbers

The graph of $f(x) = \tan^{-1} x$ is shown below:

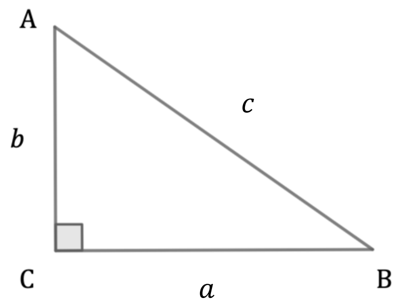


The range of $f(x) = \tan^{-1} x$ is restricted to $-\frac{\pi}{2} < y < \frac{\pi}{2}$ so there is only one output value for each input.

Section 7.9 The Reciprocal Trigonometric Functions

Looking Back 7.9

The three trigonometric ratios previously studied are as follows:



$$\sin \mathrm{A} = \frac{a}{c}$$

$$\cos \mathrm{A} = \frac{b}{c}$$

$$\tan \mathrm{A} = \frac{a}{b}$$

There are also three trigonometric functions for angle B:

$$\sin \mathrm{B} = \frac{b}{c}$$

$$\cos \mathrm{B} = \frac{a}{c}$$

$$\tan \mathrm{B} = \frac{b}{a}$$

These trigonometric functions are used to find missing sides in right triangles.

There are also three inverse trigonometric functions:

inverse sine $(\sin^{-1})$

inverse cosine $(\cos^{-1})$

inverse tangent $(\tan^{-1})$

These are used to find missing angles in right triangles as previously studied.
As has been stated, the inverse key on the calculator is used for finding missing angles in right triangles. We have learned that $\sin^{-1} x \neq \frac{1}{\sin x}$.

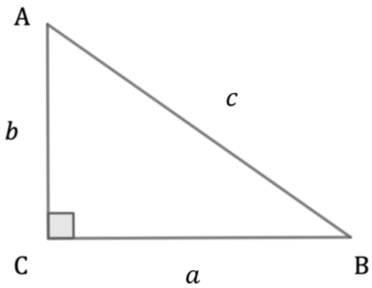
There are three trigonometric ratios that are reciprocals of the trigonometric functions:

cosecant (csc)

secant (sec)

cotangent (cot)

Looking Ahead 7.9



Example 1: If $\sin A$ is $\frac{\text{opposite}}{\text{hypotenuse}}$ then the reciprocal is $\csc A$ is $\frac{\text{hypotenuse}}{\text{opposite}}$. Find sin A and csc A for the triangle above.

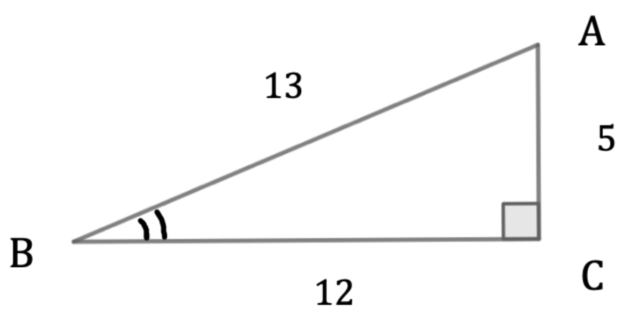
Example 2: If $\cos A$ is $\frac{\text{adjacent}}{\text{hypotenuse}}$ then the reciprocal is $\sec A$ is $\frac{\text{hypotenuse}}{\text{adjacent}}$. Find the cos A and the sec A for the triangle above.

Example 3: If $\tan A$ is $\frac{\text{opposite}}{\text{adjacent}}$ then the reciprocal is $\cot A$ is $\frac{\text{adjacent}}{\text{opposite}}$. Find the tan A and the cot A for the triangle above.

Reciprocal Trigonometric Functions:

- Secant is the reciprocal of cosine: $\sec A = \frac{1}{\cos A}$
- Cosecant is the reciprocal of sine: $\csc A = \frac{1}{\sin A}$
- Cotangent is the reciprocal of tangent: $\cot A = \frac{1}{\tan A}$

Example 4:	Find the cosecant, secant, and cotangent of angle B in the right triangle.



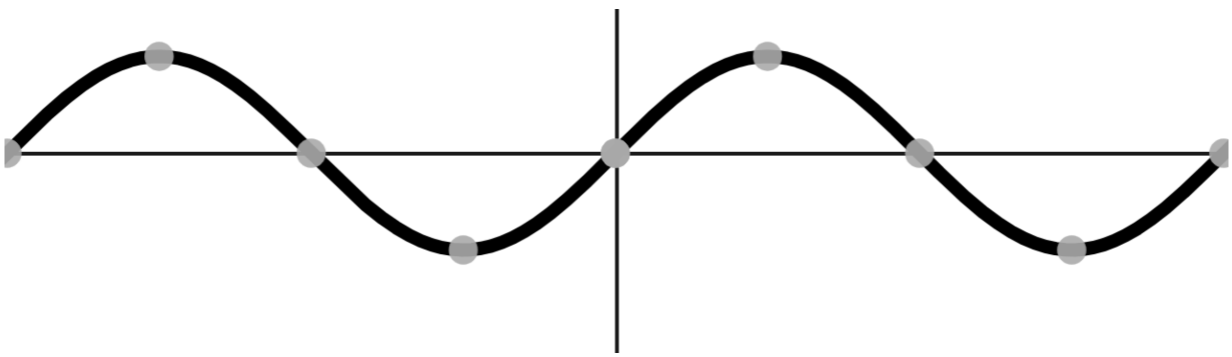
Example 5:	Find the six trigonometric functions of angle A in the triangle above in Example 4.

Section 7.10 Amplitude and Period

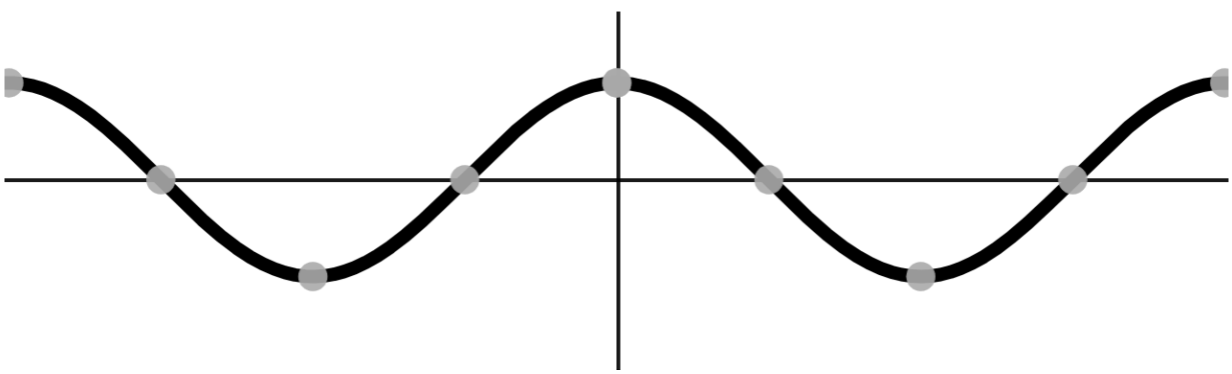
Looking Back 7.10

The trigonometric functions are called circular functions because of the unit circle. They are also called periodic functions because the graph has a pattern that repeats indefinitely.

A cycle is the shortest length before it begins to repeat. The horizontal length of the cycle is called the period. Sine and cosine have a period of 2π as was demonstrated in the previous sections where the function of the unit circle was transferred to the graph. Another property of $\sin x$ is $\sin x = \sin(x \pm 2\pi n)$ where n is any integer. The domain of sine and cosine are all real numbers $(-\infty < x < \infty)$. The range for both is $-1 \leq y \leq 1$.



If $f(x) = \sin x$ or $f(x) = \cos x$, then for any integer n, $f(x + 2\pi n) = f(x)$ for every x in the domain of f.



On the graph, θ is used to represent the angle measure in degrees. On the graph, x is used to represent the angle measure in radians.

The maximum value for $y = \sin x$ first occurs when x is $\frac{\pi}{2}$ and every period after that, or when $x = \frac{\pi}{2} + 2\pi n$ where n is an integer.

The maximum value for $y = \cos x$ first occurs when x is 2π and every period after that, or when $x = 2\pi n$ where n is an integer.

The minimum value for $y = \sin x$ occurs when $x = \frac{3\pi}{2} + 2\pi n$ where n is an integer.

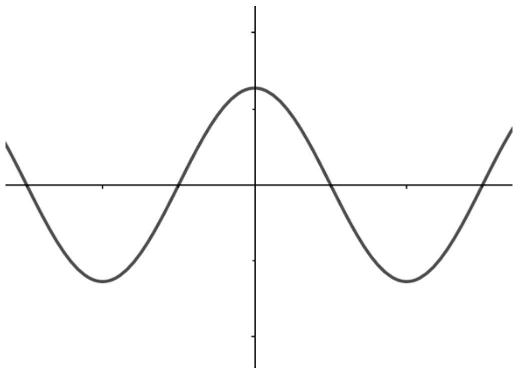
The minimum value for $y = \cos x$ occurs when $x = \pi + 2\pi n$ or $x = (2n + 1)\pi$ where n is an integer.

Looking Ahead 7.10

The critical points for $y = \sin x$ are the maximum point (at one-fourth of a period) and the minimum point (at three-fourths of the period). The x-intercepts occur at the start of the period, one-half of a period, and at one full period.

The critical points for $y = \cos x$ are the maximum points at the beginning and end of one period. The minimum point occurs at one-half a period. The x-intercepts are at one-fourth of a period and three-fourths of a period.

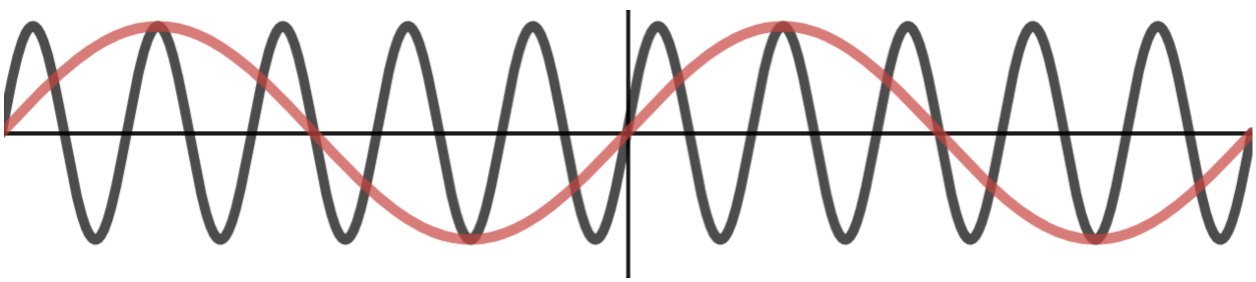
Another characteristic is the amplitude. The amplitude is the height or depth of the wave from its midline when it is at rest. It is the height measure above or below the middle of the wave for the maximum or minimum points.



$$\text{Amplitude} = \frac{|\text{Maximum - Minimum}|}{2}$$

The amplitude is the value of $|a|$ in $a \sin x = y$ or $a \cos x = y$.

The amplitude is half of the distance between the peak (maximum) and trough (minimum) of the wave. The period is the length of one full cycle before it repeats in $\sin bx = y$ or $\cos bx = y$; also.



The cycle begins at $5x = 0$ or $x = 0$, and the cycle ends at $5x = 2\pi$ or $x = \frac{2\pi}{5}$. Therefore, the period for $y = \sin bx$ is $\frac{2\pi}{|b|}$.

For any periodic function, the amplitude is $|a|$ and the period is $\frac{2\pi}{|b|}$.

Example 1: Find the amplitude and period of $y = 3\sin x$ and sketch the graph of the function.

The intercepts stay the same as the parent function, but the maximum and minimum appear as a vertical stretch on the graph.
The effect of $|a| > 1$ is a vertical stretch. The effect of $0 < |a| < 1$ is a vertical shrink. The effect of $a < 0$ is a reflection about the x-axis as we have previously seen.

Example 2: What are the amplitude and period of $y = \cos 2x$ and $y = \cos\frac{1}{2}x$. Sketch the graphs. Explain the effects of b on the graph.

The lesser period of π oscillates more frequently and the greater period of 4π oscillates less frequently over intervals of the same length; for example, 2π.
There is a horizontal shrink or compression of the graph if $|b| > 1$, and horizontal stretch is $0 \leq |b| \leq 1$.

Example 3: Plucking the string of an instrument, such as a guitar or violin, causes oscillations that can be modeled by an oscilloscope, which measures sound in Hertz. Write an equation for a sound wave modeled by the periodic function $y = a\sin bx$ where the amplitude is 2, the period is $\frac{1}{132}$, and x is in seconds. Let $a > 1$. Let $b > 1$.

Section 7.11 Graphing Trigonometric Transformations

Looking Back 7.11

You have graphed the parent function of sine, cosine, and tangent: $y = \sin x$, $y = \cos x$, $y = \tan x$.

In the previous practice problems section, you completed a table of the trigonometric functions and their reciprocals with exact values of each for the radian measurements in the unit circle.

Let us begin this section by completing a table of the critical points of each of the trigonometric functions and their reciprocals on the graph. We already investigated amplitude and period, and a and b of the periodic functions.

We have also investigated other critical points on the graph, such as x-intercepts and y-intercepts. Each cycle has five critical parts that may coincide with these points: the start of the cycle and one-fourth, one-half, and three-fourths of the way through the cycle, and the end of the cycle. The interval distance of a transformed function is $\frac{\text{period}}{4}$ because of these four critical parts that are also critical points. Below is a table of critical points for the trigonometric functions and their reciprocals.

Function	$\boldsymbol{a}$	**Amplitude**	**Period**	**Interval Space**	**Start**	$\frac{1}{4}$ **cycle**	$\frac{1}{2}$ **cycle**	$\frac{3}{4}$ **cycle**	**End**
Sine	a	$\lvert a \rvert$	$\frac{2\pi}{\lvert b \rvert}$	$\frac{\text{period}}{4}$	0	a	0	$-a$	0
Cosine	a	$\lvert a \rvert$	$\frac{2\pi}{\lvert b \rvert}$	$\frac{\text{period}}{4}$	a	0	$-a$	0	a
Tangent	a	$\emptyset$	$\frac{\pi}{\lvert b \rvert}$	$\frac{\text{period}}{4}$	0	a	Undef.	$-a$	0
Cosecant	a	$\emptyset$	$\frac{2\pi}{\lvert b \rvert}$	$\frac{\text{period}}{4}$	Undef.	a	Undef.	$-a$	Undef.
Secant	a	$\emptyset$	$\frac{2\pi}{\lvert b \rvert}$	$\frac{\text{period}}{4}$	a	Undef.	$-a$	Undef.	a
Cotangent	a	$\emptyset$	$\frac{\pi}{\lvert b \rvert}$	$\frac{\text{period}}{4}$	Undef.	a	0	$-a$	Undef.

Note: Undefined is an Asymptote.

Plotting these critical points allows us to graph one cycle of the transformed function.

Looking Ahead 7.11

Example 1:	Use critical points to graph $y = 2\sin 3x$.

$a =$

Amplitude =

Period =

Interval Space =

Start =

$\frac{1}{4}$ cycle =

$\frac{1}{2}$ cycle =

$\frac{3}{4}$ cycle =

End =

Example 2:	Use critical points to graph $y = 3\tan 4x$.

$a =$

Amplitude =

Period =

Interval Space =

Start =

$\frac{1}{4}$ cycle =

$\frac{1}{2}$ cycle =

$\frac{3}{4}$ cycle =

End =

Example 3:	Use critical points to graph $y = 5 \csc 2x$.

$a =$

Amplitude =

Period =

Interval Space =

Start =

$\frac{1}{4}$ cycle =

$\frac{1}{2}$ cycle =

$\frac{3}{4}$ cycle =

End =

Section 7.12 Phase Shifts and Vertical Shifts

Looking Back 7.12

You know how to identify the amplitude and period of a periodic function. You also know how to transform the amplitude and period of the graph based on the equation that is a transformation of the parent function.

In this section, you will explore the phase shift, which is the horizontal translation of a periodic function and the vertical translation of a periodic function. In previous functions, these have been referred to as h (horizontal shift) and k (vertical shift):

Quadratic Transformation:

$$y = a(x - h)^2 + k$$

Absolute Value Transformation:

$$y = a|x - h| + k$$

These are often referred to as c and d in trigonometric equations:

$$y = a\sin(bx + c) + d$$
$$y = a\cos(bx + c) + d$$

Like in previous graphs of equations, if $d > 0$, the graph shifts up d units; if $d < 0$, the graph shifts down d units; if $\frac{c}{b} > 0$, the graph shifts $\frac{c}{b}$ units to the left; if $\frac{c}{b} < 0$, the graph shifts $\left|\frac{c}{b}\right|$ units to the right. Let us see why.

The cycle for $y = \sin x$ begins at $bx + c = 0$ and ends at $bx + c = 2\pi$. Solve for x in each equation:

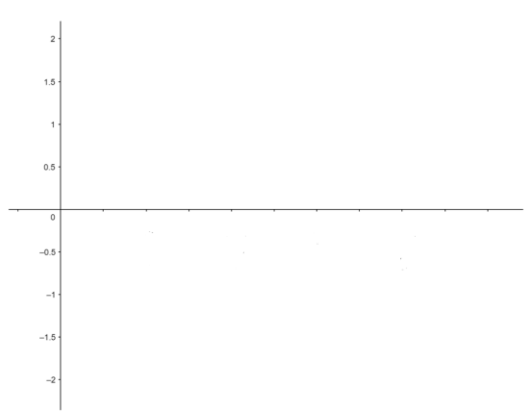
$bx + c = 0$	$bx + c = 2\pi$
$bx = -c$	$bx = 2\pi - c$
$x = -\frac{c}{b}$	$x = \frac{2\pi}{b} - \frac{c}{b}$

So, we can see that the phase shift is $-\frac{c}{b}$. If we rewrite the equation $y = f(bx + c)$ as $y = f \cdot b(x + \frac{c}{b})$, we can see the phase shift more directly.

Looking Ahead 7.12

We are going to follow the same processes as we did in the previous section, but add a few steps to investigate phase shifts and vertical shifts. We will begin with phase shifts.

Example 1:	Graph the sinusoidal function $y = 2\sin 4\left(x - \frac{1}{2}\pi\right)$.



$a =$

Amplitude =

Period =

Interval Spacing =

Horizontal Shift (at start) =

Start =

$\frac{1}{4}$ cycle =

$\frac{1}{2}$ cycle =

$\frac{3}{4}$ cycle =

End =

You can see how this can get very complicated. There is another method that may be easier for you: simply make a table of each transformation until the final x values will be at the left side of the table and the final y values will be at the right. We will investigate this method in Example 6.

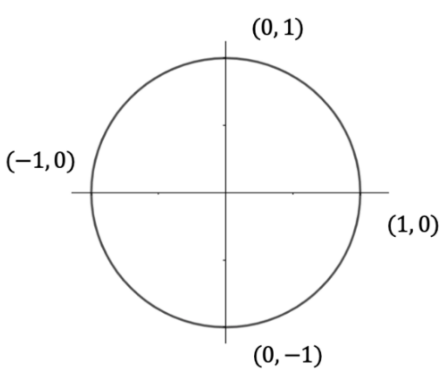
Example 2: First compare $y = \sin x$ to $y = \sin 2x$. What do you notice?

Example 3: Now compare $y = \sin x$ to $y = \sin(x + \frac{\pi}{2})$. What do you notice?

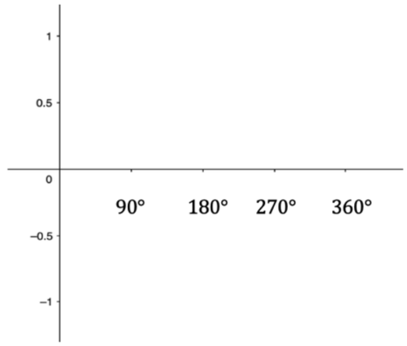
The equation $y = a \sin b\,(x + c) + d$ may also be written $y = d + a \sin b(x + c)$. Let us just look at the parameter d first.

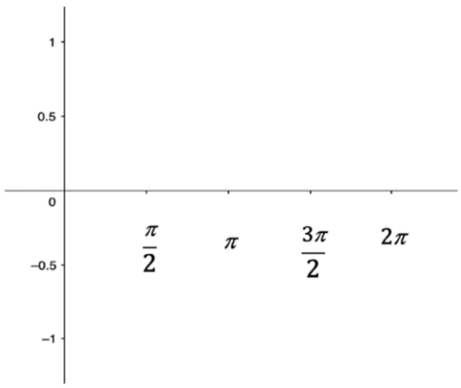
Example 4: Compare $y = \sin x$ to $y = \sin(x) + 4$ and $y = \sin(x) - 2$. Is $y = \sin(x) + 4$ the same as $y = 4 + \sin x$?

Example 5: Complete the table of critical points for the sine and cosine function using the unit circle. Graph one cycle of each.



θ	**cos θ**
0	
90°	
180°	
270°	
360°	





x	$\sin x$
0	
$\frac{\pi}{2}$	
π	
$\frac{3\pi}{2}$	
2π	

Example 6: Complete the table for $y = 10 \sin 5\,(\theta + 6) - 8$ in degrees and draw its graph.

$\mathbf{\frac{1}{5}\theta - 6}$	$\mathbf{\frac{1}{5}\theta}$	$\boldsymbol{\theta}$	$\mathbf{\sin\theta}$	$\mathbf{10\sin\theta}$	$\mathbf{10\sin\theta - 8}$
		0°	0		
		90°	1		
		180°	0		
		270°	−1		
		360°	0		

Example 7: Complete the simplified table for the equation $y = 3\sin 2\left(x - \frac{\pi}{4}\right) - 2$ using radians and graph the transformed function.

$+\frac{\pi}{4}$	$\div 2$	x	$\sin x$	$\cdot 3$	-2
		0	0		
		$\frac{\pi}{2}$	1		
		π	0		
		$\frac{3\pi}{2}$	-1		
		2π	0		

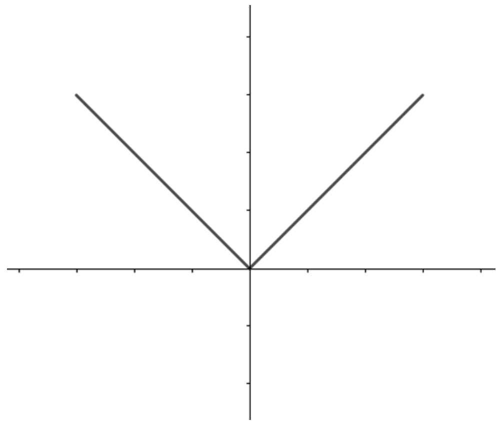
x	y

Example 8: Find the transformation from the parent function $y = \cos\theta$ to obtain the transformation $y = 3 + 5\cos\frac{1}{4}(\theta - 240°)$.

Section 7.13 Trigonometric Functions in the Real-World

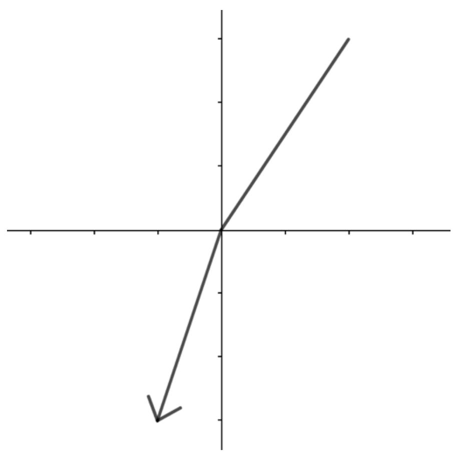
Looking Back 7.13

When light bounces off an object, such as a shiny metal, it is a reflection. The light will reflect at the same angle that it hits a surface when the surface is smooth.



The incoming ray is called the incident ray and θ_1 is the angle of incidence. The outgoing ray is called the reflected ray and θ_2 is the angle of reflection. The vertical dashed line is called the normal and it is perpendicular to the surface of the metal, which is the horizontal dashed line.

When light hits black-top it is not reflected but soaked up or absorbed. When light passes through a dense object it is bent or deflected. This happens when you put a pencil in a glass of water.



The speed of light in a vacuum is $299{,}792{,}458$ m/s, which is $c = 3 \times 10^8$ m/s when rounded up. The speed of light in a medium can be calculated by $V = \frac{c}{n}$ where V is the velocity and n is the index of refraction. If the speed of light in air is 3×10^8 m/s, what is the index of refraction for air?

If the index of refraction for water is $n = 1.33$, what is the speed of light in water? Is it faster or slower than the speed of light in air? Explain your findings.

In Algebra I, you performed an experiment using a timing fork to calculate the speed of sound through a medium. Today's experiment involves calculating the speed of light through a medium. That medium is Jell-O®.

This angle of refraction will be used along with Snell's Law to calculate the index of refraction for Jell-O®. Snell's Law is shown as follows:

$$n_1 \sin \theta_1 = n_2 \sin\theta_2$$

n_1 = index of refraction for air
n_2 = index of refraction for Jell-O®

Refraction is the change of direction of a light wave as it passes through a medium.

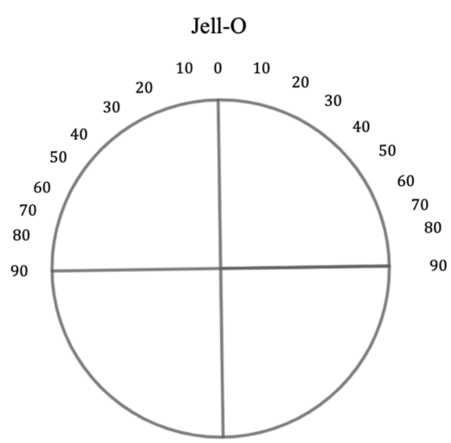
The critical angle is when the rays of light passing from a dense medium to a less dense medium (such as from Jell-O® to air) are no longer refracted but totally internally reflected, which occurs at a refraction of 90°. This critical angle is the largest angle of incidence for which refraction can still occur.

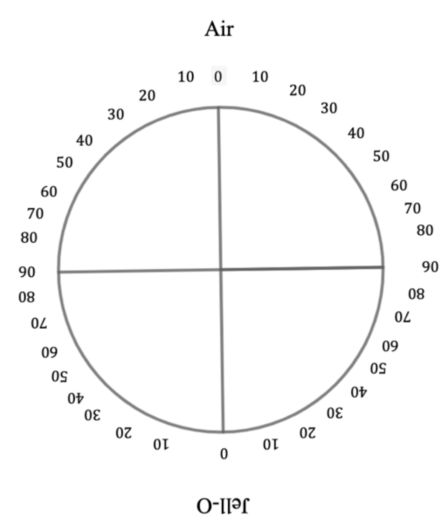
1. Before beginning the experiment, you will need to make red Jell-O® in a semicircular petri dish and let it solidify in a refrigerator.

2. On a sheet of paper, trace the petri dish. Use a protractor to make tick marks every 10°. At the top center, label it "0°" and mark 10°, 20°, 30°, etc. all the way up to 90° going to the right and do the same thing to the left of 0°.

3. Now align the petri dish with the other semicircle so a circle is formed and repeat Step 2.

4. The circle will have four 90° quadrants. Write "Jell-O®" at the top of the paper and "Air" at the bottom.





5. Place a folder behind the paper labeled "Jell-O®." Place the petri dish on top of the semi-circle labeled "Jell-O®". Wear goggles and using a red laser (do not point the laser at your eyes!), point the red laser at either 10° mark on the side of the paper labeled "Air" and you will see the laser's light go through the air and then the Jell-O® along the 10° mark, and come out the other side at the folder. The light will bend as it moves from one medium (air) to another medium (Jell-O®).

6. Point the laser at each $10°$ angle of incidence, θ_1, on the paper. Locate the angle of refraction, θ_2, through the Jell-O®. Complete the second column of the table from Air to Jell-O®. Then answer the questions for parts a) through h).

Angle of Incidence (θ_1)	**Angle of Refraction (θ_2)**	**Index of Refraction**
$10°$		
$20°$		
$30°$		
$40°$		
$50°$		
$60°$		
$70°$		

a) At each angle of incidence, does the refracted ray move closer to the normal or farther away from the normal (bend inward or outward)?

b) Use Snell's Law to calculate the index of refraction for Jell-O® for each $10°$ angle and complete Column 3 of the table. Remember that $n_1 = 1$ (the index of refraction for air). Use Column 3 to find the average index of refraction for Jell-O®. What is it?

c) Use the average index of refraction for Jell-O® calculated in part b) to find the speed of light in Jell-O®. How does this compare to the speed of light in air? Explain your findings.

d) Observe the speed of light in air compared to its index of refraction. Observe the speed of light in Jell-O® compared to its index of refraction. What is the relationship between the speed of light in a substance and the index of refraction? What the is constant of variation in the formula $n = \frac{c}{v}$?

e) Place a folder behind the paper labeled “Air.” Place the petri dish on top of the semi-circle labeled “Jell-O®”. Wear goggles and using a red laser (do not point the laser at your eyes!), point the red laser at either 10° mark on the side of the paper labeled “Jell-O®” and you will see the laser’s light go through the Jell-O® along the 10° mark and come out the other side at the folder. The light will bend as it moves from one medium (Jell-O®) to another medium (air). As the angles of the incident rays increase, does the refracted ray move closer to the 0° line (normal) or farther away from the normal (0°)?

f) Between which two angles of incidence does total internal reflection occur? (Total internal reflection occurs when the angle of incidence is greater than the critical angle.)

g) Use Snell’s Law to calculate the critical angle (θ_1) from Jell-O® to air and see if it is between the two angles of incidence you observed in part f). Use n_1 (this time it is the index of refraction for Jell-O®), and n_2 (this time it is the index of refraction for air). Let θ_2 be 90° to calculate θ_1 (the critical angle for Jell-O®). The critical angle of the Jell-O® is the angle of incidence at which the refracted angle is 90°. This occurs when the speed of light increases from one substance into another.

h) The index of refraction is a dimensionless number. What information does the index yield? In other words, how would you define the index of refraction?