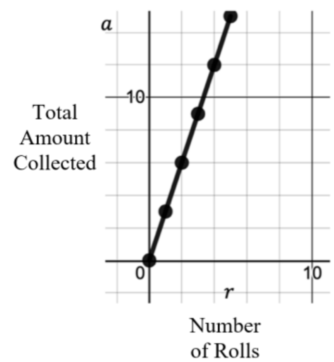
Pre-Calculus and Calculus Module 6 Rates of Change

Section 6.1 Constant Rates of Change

Looking Back 6.1

Rates of change are a foundational mathematical concept. When we first investigated rates of change in Pre-Algebra, the slope (inclination) of a linear equation was identified as the rate of change.

The "Beyond the Walls" outreach team sells gluten free rolls at $3.00 each to raise funds for Children's Heart Hospital. If we let a be the total amount collected and r be the amount per roll, the equation to represent total amount collected on all sales is $a = 3r$. If 1 roll sells, $3 is collected. If 20 rolls sell, $60 is collected. The graph of this equation is linear; it is the direct variation $y = mx$. The y-intercept is 0; 0 rolls sold means $0 collected.



r	a

The constant rate of change is $3 per roll. If the graph is not linear, the rate of change is not constant. Calculus allows us to evaluate changing rates of change (those that are not constant). Limits help us evaluate instantaneous rates of change at any given point in time.

The slope triangles represent the constant rate of change: $m = \frac{y_2-y_1}{x_2-x_1}$ and $\frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x}$; $\frac{\Delta y}{\Delta x}$ is said to be the change in y over the change in x.

If we look at the slope triangles from a trigonometric viewpoint and let θ be the angle formed counterclockwise from the r-axis toward the a-axis, tan θ is equal to $\frac{3}{1}$.

$$\begin{aligned} \text{Tan}\,\theta &= 3 \\ \theta &= \tan^{-1}(3) \\ \theta &\approx 72^\circ \end{aligned}$$

When you use your calculator, make sure it is set to "degree" mode and not "radian" mode. Checking $\tan 72^\circ \approx \frac{3}{1}$ gives the slope, which is equal to the tangent. Therefore, the slope is 3 and the angle of inclination is approximately 72°.

Let us investigate several tables for the problem and see if a pattern emerges that is evident.

r	a
1	\$3
2	\$6
3	\$9
4	\$12

r	a
1	\$3
3	\$9
5	\$15
7	\$21

r	a
1	\$3
4	\$12
7	\$21
10	\$30

A pattern is evident. As x increases by 1 (the same constant) each time, y increases by 3 (the same constant) each time. If we look at other tables of the same function, as x increases by 2 each time, y increases by 6 each time; or as x increases by 3 each time, y increases by 9 each time. These all represent the same slope: $\frac{\Delta y}{\Delta x} = \frac{3}{1} = \frac{6}{2} = \frac{9}{3}$. This property is called the "add-add property of linear functions" because the same constant is being added to x each time, and the same constant is being added to y each time. However, the constant added to x each time is not the same as the constant being added to y each time, although it can be the same as in the parent linear function $y = x$.

Looking Ahead 6.1

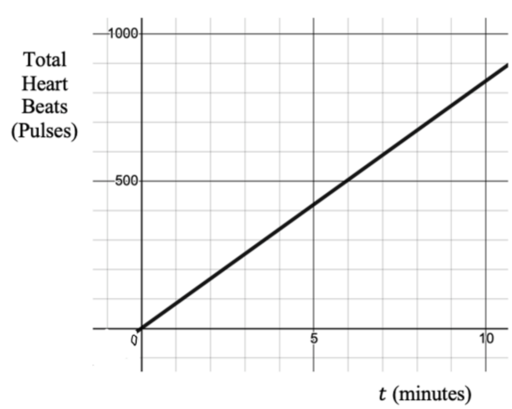
Sit still for approximately 15 minutes while keeping time on a stopwatch. After the 15 minutes is up, you will measure your resting heart rate by measuring beats per minute.

Put two fingers on your wrist until you find a pulse (you should feel a throb). Start the stopwatch and stop it after 15 seconds. Count and record the number of pulses in the 15 seconds. Multiply the number you get by 4 to find the number of pulses in 60 seconds (1 minute). Make a table of your total heart beats for each minute up to 10 minutes. You will use this table in the Practice Problems.

Example 1: Shelly counted her pulse for 15 seconds. She counted 21 beats. Find beats per minute for Shelly's resting heart rate (she was sitting down). Make a table and graph to show the total beats per minute over a 10-minute interval.

$$15 \text{ seconds } \cdot 4 = 60 \text{ seconds}$$
$$21 \text{ beats } \cdot 4 = 84 \text{ beats per minute (BPM)}$$

Time (min.)	**Total Heart Beats**
1	84
2	168
3	252
4	336
5	420
6	504
7	588
8	672
9	756
10	840



a) What is the slope?

b) Without measuring the angle of inclination (θ), what would the tangent of θ be if the scales of the $x$and y axes had the same scale? (Note: If you did measure the angle of inclination with a protractor, the angle would be different because the x-scale is much larger than the y-scale.)

c) What is the rate of change?

The total beats per minute would increase at a fairly constant rate barring any unforeseen irregularities or activities.

I hope you are making the connections here. The $\frac{\Delta y}{\Delta x}$ is the same as the slope and the tangent of the slope triangles, and is the rate of change.

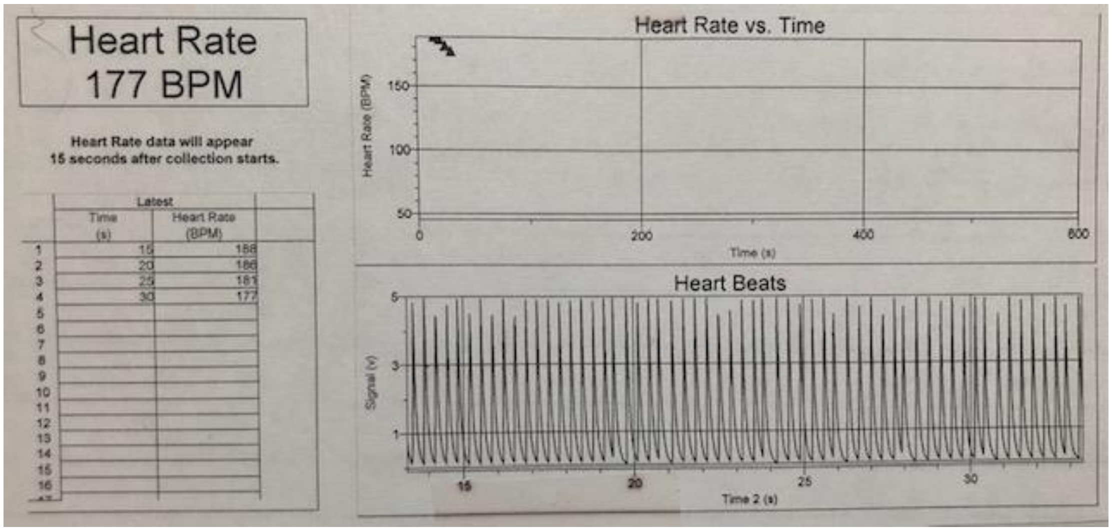
Example 2: Examine the table and graph for Faith's heart rate. Find the rate of change and answer the questions below.

a) Is it a constant rate of change?

b) Is it increasing or decreasing?

c) What do you think Faith is doing?

Note that the x-axis represents seconds, not minutes as in our previous graph. Moreover, the heart monitor is recording the heart rate in beats per minute on the y-axis, not total heart beats as in Example 1.



Calculating the rate of change does not give us a constant rate. The change of rate…

From 15 seconds to 20 seconds:

$$\frac{\Delta y_1}{\Delta x_1} \qquad \frac{186-188}{20-15} = \frac{-2}{5} = -0.4 \text{ BPM}/s$$

From 20 seconds to 25 seconds:

$$\frac{\Delta y_2}{\Delta x_2} \qquad \frac{181-186}{25-20} = \frac{-5}{5} = -1 \text{ BPM}/s$$

From 25 seconds to 30 seconds:

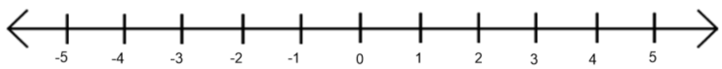
$$\frac{\Delta y_3}{\Delta x_3} \qquad \frac{177-181}{30-25} = -\frac{4}{5} = -0.8 \text{ BPM}/s$$

Though it does not make sense to have negative beats per minutes per second, that simply tells us the heart rate is decreasing. If we find the rate of change over the entire interval of time:

$$\frac{\Delta y}{\Delta x} = \frac{177-188}{30-15} = \frac{-11}{15} \approx 0.73 \text{ BPM}/s.$$

It appears our rate of change is changing. That will be the topic of the next section and a major focus of Calculus. For now, we will introduce the similarities and differences between position, displacement, distance, speed and velocity for our final example.

Example 3: A ball starts at 3 and rolls to 2 at 1 second, 0 at 2 seconds, -2 at 3 seconds, and sits at -1 for 4, 5, and 6, seconds, then rolls steadily toward 1 and gets there at 10 seconds. Find the position, displacement, and distance of the ball from start to finish and draw the position-time graph.



Scalars

Distance: the amount of space between two things.

If a change of direction occurs, total distance will be greater than displacement.

$$d = |\Delta x_1| + |\Delta x_2|$$

Position: the location of an object or person from one point in time to another.

$$x = s(t)$$

$$\text{Average Speed} = \frac{\text{distance}}{\text{time}}$$

$$|v| = \text{speed}$$

Vectors

Direction matters so vectors can be negative.

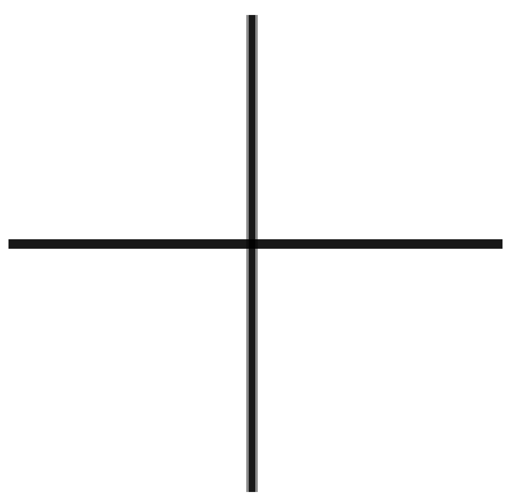
Displacement: the difference between an object's initial position and its final position.

$$\Delta x = x_{\mathrm{F}} - x_{\mathrm{I}}$$

Velocity: the rate of change in displacement over elapsed time.

$$\text{Velocity} = \frac{\text{change in position}}{\text{change in time}}$$

$$v(t) = \frac{s(b) - s(a)}{b - a}$$



Section 6.2 Changing Rates of Change

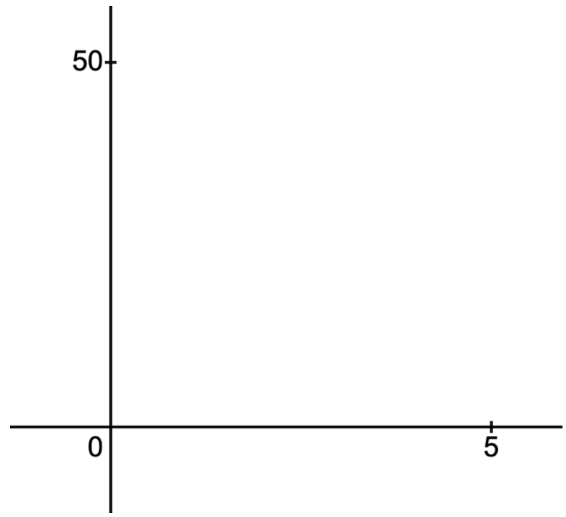
Looking Back 6.2

Some parasites move from the liver to the bloodstream in lethal malaria cases. Moreover, they often move at an exponential rate. The exponential rate is called a growth factor in the equation.

Let us assume, for our purposes, that there are 3 parasites in the liver that double in quantity approximately every day. This is a case of exponential growth and is modeled by the equation $f(x) = 3 \cdot 2^x$. Because the number of parasites is doubling, 2 is the growth factor. This is not the same as the rate of change. The growth rate is 100%.

A table and graph for the function $f(x) = 3 \cdot 2^x$ shows that as x is changing at a constant rate, $f(x)$ is not.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$



Whenever a constant amount is added to x, $f(x)$ doubles each time due to the growth factor. This is called the add-multiply pattern of exponential functions that introduced in Algebra 2 and proved in Geometry and Trigonometry.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$

As 1 is being added to x each time, $f(x)$ is being multiplied by 2. The rate of change is smallest first because the numbers being multiplied are small, but the rate of growth seems to increase over time for exponential functions. Again, a table of whole number values for x and the corresponding $f(x)$ demonstrates that as x increases at a constant value, $f(x)$ is being multiplied by another value that stays the same.

$\boldsymbol{x}$	$\boldsymbol{f(x)}$

$\boldsymbol{x}$	$\boldsymbol{f(x)}$

Looking Ahead 6.2

Exponential functions seem to be changing at a changing rate. Let us investigate power functions. We will start with a cubic function.

Example 1: Using the power function $g(x) = 5x^3$, generate a table and graph and determine if adding a constant to x generates a corresponding pattern for $g(x)$.

$\boldsymbol{x}$	$\boldsymbol{g(x)}$

$\boldsymbol{x}$	$\boldsymbol{g(x)}$

There does not seem to be an add-add pattern or an add-multiply pattern. There does seem to be a multiply-multiply pattern.

$\boldsymbol{x}$	$\boldsymbol{g(x)}$

Example 2: The function $h(x)$ has the values $h(4) = 20$ and $h(9) = 35$. Find the equation if it is linear. Use what you know about the add-add property of linear functions to solve the problem.

$\boldsymbol{x}$	$\boldsymbol{h(x)}$

Example 3: Using the table below, find the equation for the power function.

$\boldsymbol{x}$	$\boldsymbol{g(x)}$
2	48
4	768
8	12,288
16	196,608

Example 4: If x is tripled in each of the situations below, describe how y is affected. Assume x is positive.

a) y varies directly with the third power of x

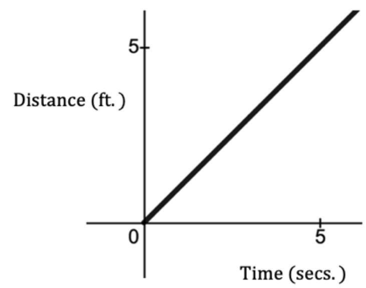
b) y varies inversely with the square root of x

Section 6.3 Distance-Time Graphs

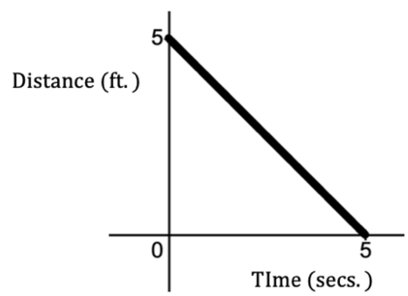
Looking Back 6.3

Motion detectors can be used to create distance-time graphs. These use sound radar to measure the distance an object is in relation to the motion detector.

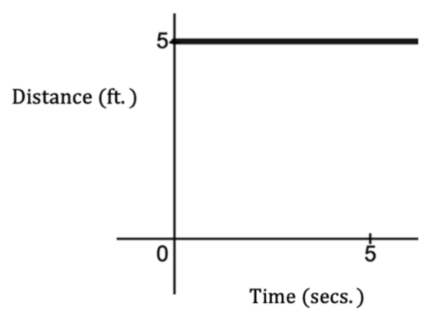
As the object gets farther away from the motion detector, the distance increases.



As the object gets closer to the motion detector, the distance decreases.



If an object does not move, the distance from the motion detector stays the same.



The object can be a person, a ball, a plane, or a car. It can be anything that moves and has motion.

Distance-time graphs are used to analyze motion. Much of physics concerns itself with where an object is located, how fast it is moving, and how its motion is changing. Where an object is located is its position or distance from a fixed point.

If a car travels at 30 m.p.h. and leaves from home, it will be 90 miles from home after 3 hours.

$$\begin{cases} d = rt \text{ is linear} \\ y = mx \text{ is linear} \end{cases}$$

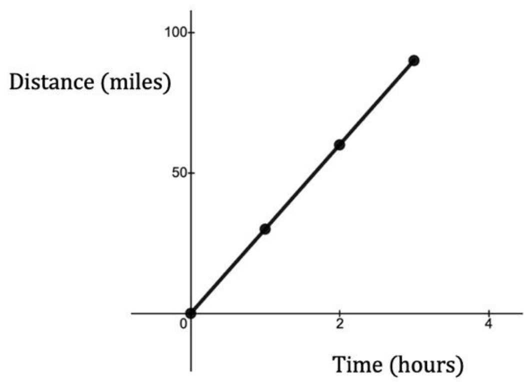
The distance is represented on a graph by the y-axis. The time is represented on a graph by the x-axis. Hence the name, distance-time graph. Can we determine the speed of an object from a distance-time graph? Yes:

$$\frac{d}{t} = r$$

$$\frac{\Delta y}{\Delta x} = m$$

$$\frac{d_2 - d_1}{t_2 - t_1} = r$$

The rate of change of the distance-time graph of a car, for example, is its speed.



$d(t) = 30t$

$d(1) = 30$

$d(2) = 60$

$d(3) = 90$

Looking Ahead 6.3

In Physics, you learn that the distance an object travels is measured by how far it moves and does not contain directional information; it is a scalar quantity. However, displacement, although it can be equal to distance, can be a different direction. Displacement contains directional information and is a vector quantity.

Speed is the time rate of change of the distance traveled by an object:

$$\text{Speed} = \frac{|d_2 - d_1|}{t_2 - t_1} = \frac{\Delta d}{\Delta t}$$

On the graph, $\frac{\Delta y}{\Delta x} = \frac{\Delta d}{\Delta t}$.

To calculate speed, we divide the total distance an object travels by the total time is travels, regardless of the direction it travels.

Displacement is a change in the position of an object and has both magnitude and direction.

$$\text{Velocity} = \frac{d_2 - d_1}{t_2 - t_1} = \frac{\Delta d}{\Delta t}$$

In speed, Δd represents the change in distance. In velocity, Δd represents the change in displacement. Speed is how fast an object is moving. Velocity is how fast an object is moving and in what direction.

Example 1: Each linear function models the motion of an object. Answer the questions that follow the linear functions.

a) $d(t) = 2.1t + 1.5$

b) $d(t) = -2.1t + 0.5$

c) $d(t) = 2.3t + 2.5$

d) $d(t) = 2.1t + 1.7$

1. Which objects are traveling at the same velocity?

2. Which objects are traveling at the same speed and same direction?

3. Which object moves the fastest?

4. Which object starts closest to the motion detector?

Example 2: Graph the distance-time data for the model rocket using the axes on the next page and answer the questions that follow.

Flight of a Model Rocket:

Time (secs.)	**Distance (m.)**
0	0
1	24.2
2	48.8
3	85.9
4	140.2
5	155
6	194.6
7	198.2
8	202.5
9	198.2
10	174.6
11	165.1
12	144.3
13	136.2
14	130

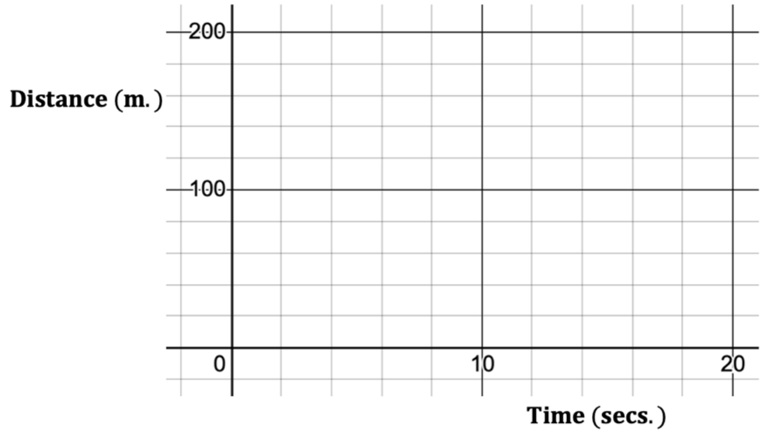
Use the coordinates to answer the following questions:

a) When does the rocket take off?

b) When does the rocket reach its maximum height?

c) When does the rocket's engine burn out?

d) When does the rocket's parachute open?



Example 3:	Use the table of values for the flight of the rocket from Example 2 to answer the questions below.

1. What is the interval of ascent?

2. What is the interval of descent?

3. What is the average speed of the model rocket in the interval $[7, 9]$?

4. What is the average velocity of the model rocket in the interval $[7, 9]$?

Section 6.4 Slope and the Secant Line

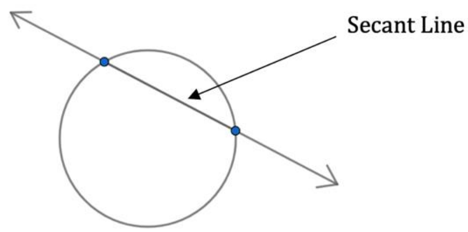
Looking Back 6.4

When a function is linear, the slope is constant; the rate of change stays the same. When a function is non-linear, the slope varies; the rate of change changes.

Motion can be modeled by functions. The slope is the rate of change. The slope then represents speed or velocity when a function is used to model motion. This is because speed is the change of distance with respect to time and velocity is the time rate of change of displacement of an object in a particular direction.

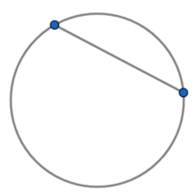
To find the slope, we find $\frac{\Delta y}{\Delta x}$ (the change in y over the change in x). To find $\frac{\Delta y}{\Delta x}$, we must use two points on the curve of the function given it is non-linear.

A line that passes through at least two points on a curve is a secant line.



This is a secant line of a circle. The secant line of the circle is any which intersects the circle at two points.

In Geometry and Trigonometry, we learned that a line segment on the interior of a circle that has both endpoints on the circle is called a chord. A secant extends through the circle.

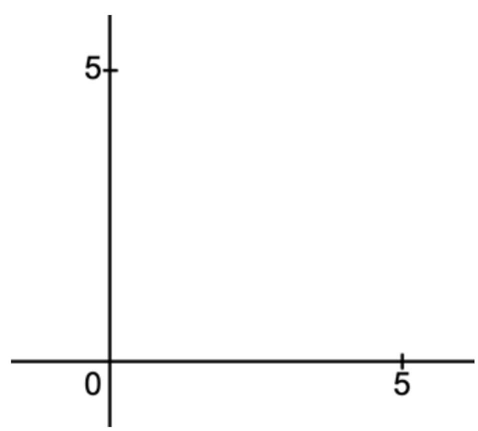


By finding the slope of a secant line, we can find the average rate of change.

Looking Ahead 6.4

Example 1:	Sketch the graph of the function $f(x) = -x^2 + 3x + 2$ and answer the questions that follow.

1) On what interval does the function appear to be increasing?



2) On what interval does the function appear to be decreasing?

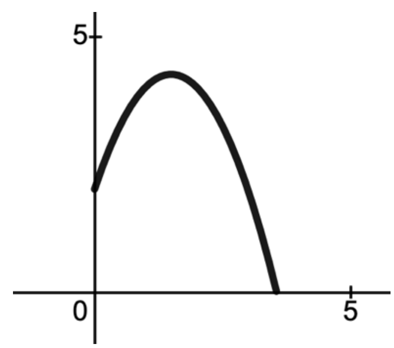
3) On what intervals is the slope positive?

4) On what intervals is the slope negative?

5) What seems to be the relationship between the two?

Example 2: Using the function $f(x) = -x^2 + 3x + 2$ and the sketch of its graph, find the slope of the secant line on the intervals that follow.

a) $[0, 2]$

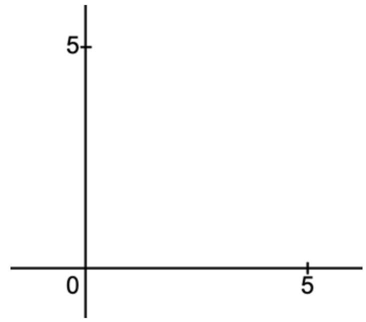


b) $[0, 1]$

c) $[0, 0.5]$

Example 3: Below is a table of data for a toy push car that came into the range of a motion detector between 1 and 4 seconds of travel. Graph the data and answer the questions that follow.

Time x (secs.)	**Distance y (m.)**
1	3.5
2	1.2
3	0.7
4	0.3



1) Is the push car moving towards the motion detector or away from it?

2) Is the push car speeding up or slowing down?

3) Find the average speed of the push car between 1 and 4 seconds.

4) Can we determine if the speed of the push car is consistently decreasing?

5) How does the average velocity of the push car compare to the average speed of the push car?

Section 6.5 Average Rates of Change and Velocity

Looking Back 6.5

The slope of a function gives us a rate of change. The slope of a secant line on the graph of the function gives us an average rate of change. As the secant line gets smaller and smaller and closer to a point, we get a more accurate rate of change at that point. It is as if we are using our calculators to zoom in on a point on the function.

We have been estimating slopes when given discrete data. In this section, we will investigate functions to see if we can come up with an algorithm that would work to find the slope at a given point for all functions.

Looking Ahead 6.5

Example 1: For the following functions, determine if the average rate of change over the interval given is positive or negative.

a) $f(x) = x^2$ over the interval $[3, 4]$

b) $g(x) = \cos 2x$ over the interval $[0.1, 1]$

c) $h(x) = \log_2 x$ over the interval $[0.1, 1]$

d) $j(x) = (\frac{1}{2})^x$ over the interval $[-4, -3]$

Example 2: Let us revisit the toy push car from the previous section and say it travels at a distance $3.1t^2$ meters in the first t seconds after it starts. Calculate the average velocity of the toy push car over the time interval given and determine its exact velocity after 2 seconds.

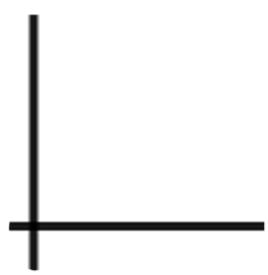
a) $[2, 3]$

b) $[2, 2.9]$

c) $[2, 2.01]$

d) $[2, 2.001]$

Example 3: Represent the average rate of change graphically and answer the questions that follow using function notation.



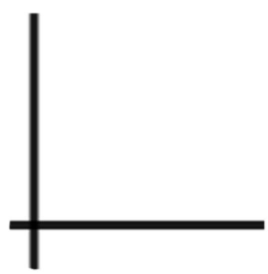
1) What is an expression that represents the change in the inputs (Δx)?

2) What is an expression that represents the change in the outputs $(\Delta f(x))$?

3) The average rate of change for $f(x)$ over the interval $[x_1, x_2]$ is the slope of the secant line between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Write an expression for this average rate of change for functions $(\frac{\Delta f(x)}{\Delta x})$.

We have seen this notation in the previous Practice Problems section, and we have seen the examples in this section use discrete data and numerical calculations. Now we have seen this notation graphically for functions. The following will help you investigate rates of change algebraically and derive an algorithm that works for all functions.

Example 4: Let P be the point $(a, f(a))$. Let Q be the point $(a+h, f(a+h))$ where $\Delta x = h$. Represent graphically how the slope of the secant line becomes the slope of the tangent line as h approaches 0.



The limit is the slope of the curve and the tangent to the curve at the point $x = a$. The limit is the functions rate of change with respect to x at the point $x = a$. This was a major contribution to Calculus by Pierre de Fermat to Calculus in 1629 for any function $y = f(x)$ where a limit exists. We have been doing this previously, but now we know why it works.

Section 6.6 Average Rates of Change

Looking Back 6.6

In Problem 5 and Problem 6 of the previous practice problems section, we found the average velocity for a toy push car over time t. We looked at the interval of time from 4 to 4.1 seconds using the distance function $d(t) = 10.17t^2$ to find its average velocity. We then investigated the average velocity using smaller time intervals.

We also investigated the average rate of change graphically in the last section. In doing this, we found the average rate of change for the function on the interval $[x_1, x_2]$ to be the slope of the line through the points on the curve that correspond to the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

In this section, we will further explore average velocity to find a general algorithm that can be used to find average rates of change, including velocity, for any function over any interval.

Looking Ahead 6.6

Let us generalize steps used in the toy push car problem using the distance function $d(t) = 10.17t^2$ for Example 1.

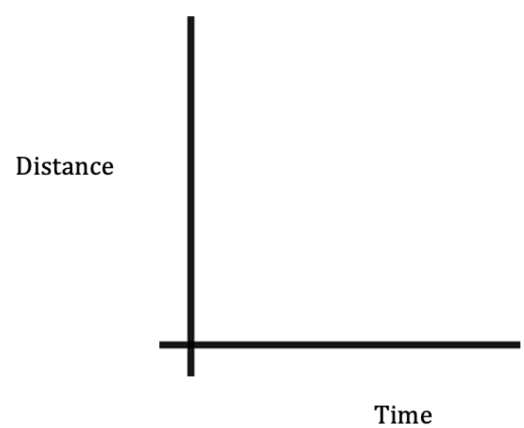
Example 1: Let h be any period of time the car travels after 4 seconds to answer the following questions.

1) What is the change in time from time over the interval $[4, 4 + h]$?

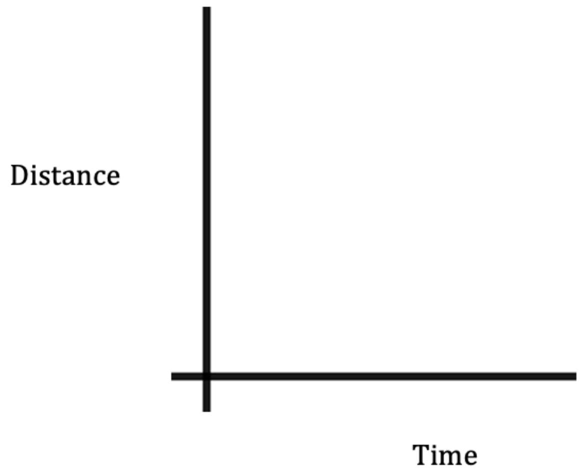
2) What is the change in distance over the interval $[4, 4 + h]$?

3) What is the average velocity of the toy push car over the same interval $[4, 4 + h]$?

Let us investigate the situation illustrated graphically.



Example 2: Let us generalize Example 1 even more and find the algorithm for a toy push car that travels any interval of time h after t seconds. We will answer the same three questions from Example 1 but using the interval of time $[t, t + h]$ illustrated graphically below.



1) What is the change in time from $t = h$ to $t = t + h$?

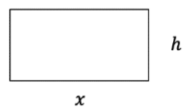
2) What is the change in distance over the interval $[t, t + h]$?

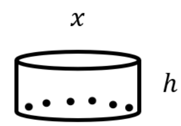
3) What is the average velocity of the toy push car over the same interval $[t, t + h]$?

Example 3: In Geometry and Trigonometry, we constructed kaleidoscopes physically and with the use of technology. Let us revisit the kaleidoscope now.

A piece of cardboard is used to make the tube having a perimeter of 24 inches. Let the length be x and the width be h. The tube can be rolled up to make a smaller and smaller tube or it can be rolled so the right h will meet the left side and the tube will have maximum volume.

Let us find the maximum dimensions of the tube that give it maximum volume by completing the steps below.





1) Find the equation of the volume of the tube in terms of x and h. (Hint: Use $V = \pi r^{2}h$ and solve for r in terms of x first.)

2) Find the volume of the tube in terms of x only. (Hint: In order to find h in terms of x, use the given information: $\mathrm{P} = 24$ in.)

3) Graph the volume function on your calculator. Change the window settings to…

$$X \text{ Min } -10 \qquad Y \text{ Min } -10$$
$$X \text{ Max } 26 \qquad Y \text{ Max } 26$$

Use the menu functions to analyze the graph and find the width of the tube that maximizes volume.

4) Now that we know the optimal width, find the maximum volume of the tube using the formula we found in Step 2.

5) Find the height of the tube that maximizes its volume. Use the formula we found in Step 2 (under the Hint section) when we solved for h in terms of x. In the following section, we will be finding the slope at the maximum height and hopefully see another important use for average rates of change and limits. For now, let us look at the solution.

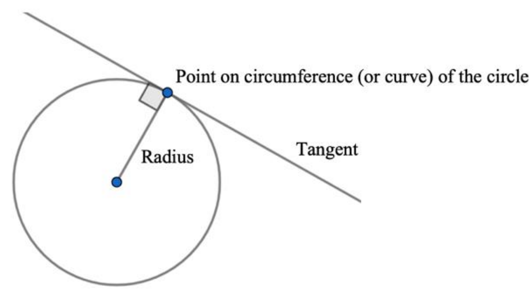
Section 6.7 Slope and the Tangent Line

Looking Back 6.7

In the previous sections, we found the velocity over the first several seconds of travel of a toy push car using the distance function. The car traveled in one direction away from the motion detector so the change in distance was the change in position and speed was equal to velocity.

In Section 6.4 the secant line was defined as a line that passes through at least two points on a curve. The secant line was used to find the slope of the curved line so the average rate of change could be found. As the two points on the curve came closer and closer together the secant line appeared to approach a limit. In this section, we will see just what that limit is.

As the two points get closer together, the secant line becomes a tangent line. A tangent line is the line that touches a curve at one point. On a circle, the secant line cuts across it as a chord. A tangent line touches the circle at one point on the curve and does not cut across it. It is perpendicular to the radius at that point on the circle (as we learned in Geometry and Trigonometry).

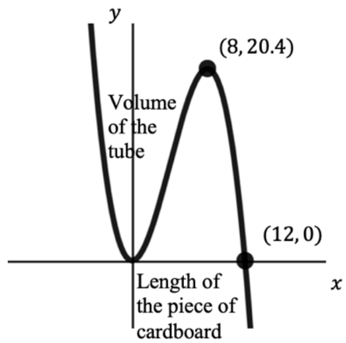


How can you find the slope of a tangent line when it is just one point, and two points are needed to calculate slope? Let us find out.

Looking Ahead 6.7

Let us revisit the tube from the previous section that was used to build the kaleidoscope. The formula derived for the volume of the tube was $V = \frac{12x^2 - x^3}{4\pi}$. The function for volume is in terms of the length (x): $f(x) = \frac{12x^2 - x^3}{4\pi}$.

The graph of the function increased on the interval $[0, 8]$ and decreased on the interval $[8, 12]$.



The maximum volume of the kaleidoscope tube is 20.4 cm.^{3} when the cardboard piece is 8 cm. long.

Example 1: Find the slope over the function intervals as x approaches 8 from the left.

a) $[7.9, 8]$

b) $[7.99, 8]$

c) $[7.999, 8]$

Example 2: Find the slope over the function intervals as x approaches 8 from the right.

a) $[8, 8.1]$

b) $[8, 8.01]$

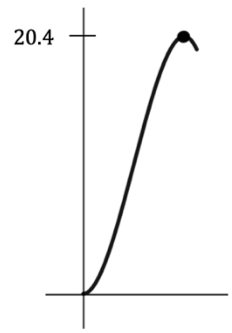
c) $[8, 8.001]$

The slopes in a)-c) of Example 1 are almost identical to the slopes the same distances away in a)-c) of Example 2.

The only difference is the sign of the slope. In Example 1, the slopes are positive because the graph is increasing to the left of 8 or over the interval $[0, 8]$. In Example 2, the slopes are negative because the graph is decreasing to the right of 8 or over the interval $[8, 12]$.

Example 3: Define the relationship between the slope of the secant lines near $x = 8$, and the slope of the tangent line at the maximum point $x = 8$.

If we zoom in over the interval $[7.9, 8.0]$, we can see the secant lines that give us the slopes in a)-c) of Example 1.



Section 6.8 Instantaneous Rates of Change and Velocity

Looking Back 6.8

Police officers uses radar guns or laser speed guns to collect distance-time data of cars that are moving. The velocity of the car at a particular instant in time is called instantaneous velocity. These devices can accurately predict instantaneous velocity for the speed of a car at the moment it passes a police officer.

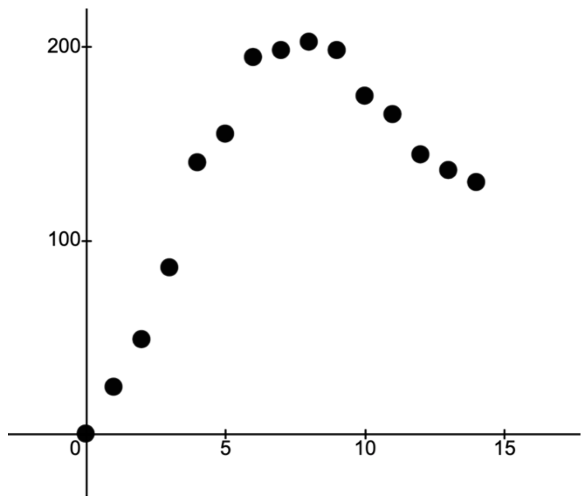
In Section 6.3, we explored the flight of a model rocket. Let us revisit that problem and learn about instantaneous velocity.

Looking Ahead 6.8

The rocket is falling over the interval $[8, 14]$ according to the table and graph. Use the information given to complete the examples that follow.

Flight of a model rocket:

Time (sec.)	Distance (m.)
0	0
1	24.2
2	48.8
3	85.9
4	140.2
5	155
6	194.6
7	198.2
8	202.5
9	198.2
10	174.6
11	165.1
12	144.3
13	136.2
14	130



Example 1: 1) Determine the average velocity for the fall over the interval 8 to 14 seconds.

2) Determine the average velocity of the fall over the interval $[8, 11]$.

3) Which average velocity do you think is closer to the instantaneous velocity at 9 seconds, the answer to 1) or 2) in this example? Why?

The instantaneous velocity can be determined by taking the average velocity over smaller and smaller intervals and the point for which you are finding the instantaneous velocity must be included in the interval.

Example 2: Find an approximation for the instantaneous velocity of the rocket at 5.5 seconds. Use these intervals and determine which is the best approximation.

a) $[4, 8]$

b) $[5, 7]$

c) $[5, 6]$

The most accurate approximation of instantaneous velocity is the answer to c) as it is the smallest interval over which 5.5 is included.

We do not have an equation that models the curve although we do have a graph. If we had an equation, we would have continuous data and be able to find the average velocity at 5.499 seconds and 5.501 seconds to get a more accurate approximation.

We found the slopes of the secant lines between the points. The interval $[5, 6]$ is the closest we can get to the tangent line given the table data.

Section 6.9 Instantaneous Rates of Change for Functions

Looking Back 6.9

Let us summarize what we have learned about rates of change so far. The secant line of a curve is a line that passes through any two points on the curve. The slope of the secant line is the average rate of change of the curve.

A tangent line is a line that touches the curve at one point only. The limit of average rate of change (the slopes of the secant lines), gives us the instantaneous rate of change of the curve (the slope of the tangent line).

Let us investigate how we can find the slope of a line at one point.

Looking Ahead 6.9

If the average rate of change for a function over the interval $[x, x+h]$ is shown below:

$$\frac{f(x+h)-f(x)}{(x-h)-x} = \frac{f(x+h)-f(x)}{h}$$

Therefore, the instantaneous rate of change is the limit of the average rate of change as h approaches 0.

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

Remember, h is the same as Δx because it is the change in x. As the interval gets smaller and smaller and closer to the given point on either side, the change in x is getting smaller and smaller and approaching 0.

A limit, L, of $f(x)$ as x approaches c, where c is a fixed point, is the one number that $f(x)$ stays close to as x gets arbitrarily close to c but not equal to c.

This limit is called the derivative of the function. The first derivative may be written "$f'(x)$" and read "f prime x."

The derivative of $f(x)$ at $x = c$ is the instantaneous rate of change of $f(x)$ with respect to x at the point when $x = c$.

On a graph, this is found from calculating the slope of the line tangent to the graph at $x = c$. Algebraically, this is found by taking the limit of the average rate of change over the interval $[x, c]$ as x approaches c. We will investigate this is Example 4. For now, we will define our derivative over the interval $[x, x+h]$ as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Example 1:	Let $d(t) = 3t^2$. Find $\frac{d(3+h)-d(3)}{(3+h)-3}$. Find the instantaneous rate of change over the interval.

We can find the instantaneous rate of change by first finding the average rate of change and taking the limit of that as h approaches 0.

Function notation may be d with $d(t)$, but the graphing calculator uses x with $f(x)$.

It is important to note that the derivative of a function $f'(x)$ is itself another function. It will take on some value when a particular input value x is specified, but $f'(x)$ is as much a function as $f(x)$ is, provided the limit which defines it exists. In this sense, derivation is a process which transforms one function into another that shares a special relationship with the original function. In particular, the derivative of a function gives the slope of the tangent line to the original function at every point in its domain.

Example 2: Let $f(x)$ be $x^2 + 3x$. Find $\frac{f(4+h)-f(4)}{(4+h)-4}$. Find the instantaneous rate of change over the interval.

Example 3: Find the average rate of change $r(x)$ of a function f over the interval that starts at $x = c$. This would be the change from c to some point x over the interval $[c, x]$. It is the change in y-values divided by the change in the corresponding x-values. Draw a graph below.

Example 4: Find the instantaneous rate of change $f'(c)$ of a function f at $x = c$ using the interval that starts at $x = c$, both algebraically and graphically.

Algebraically:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

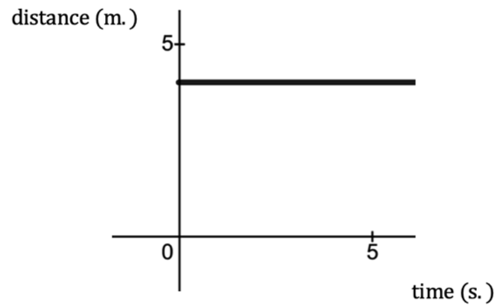
This is the formal definition of the derivative of a function at a point. The derivative of the function f at $x = c$ is the limit of the average rates of change of $f(x)$ as x approaches c.

Graphically:

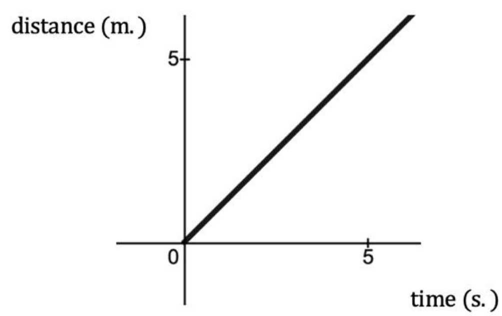
Section 6.10 Velocity and Position Graphs

Looking Back 6.10

In Section 6.3, we investigated distance-time graphs. These graphs give us information about how far an object has moved with time. The steeper the graph, the faster the motion (more distance is being covered in less time). A horizontal line on the motion detector means there is no motion. The object is not moving; it is at rest. A downward slope means the object is returning to the start.



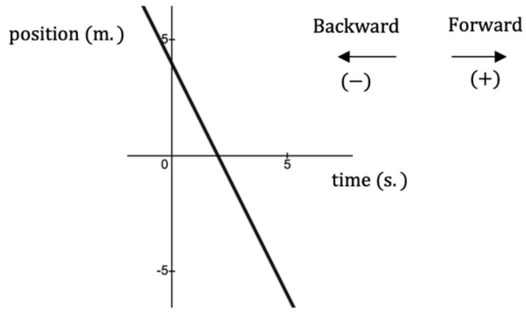
At 1 second, the object has traveled 4 meters. At 4 seconds, the object has still only traveled 4 meters. This means the object has not moved.



At 1 second, the object has traveled 1 meter. At 4 seconds, the object has traveled 4 meters. The object is traveling at 1 m/s. The slope of the graph is 1. Because the graph is linear, there is no tangent line as there is no curve. Any other line either intersects once with the curve or is identical to it, so secant and tangent are not precisely defined for a linear function. They are also meaningless since the slope in a linear function is readily available. The slope of the distance graph is velocity.

Let us investigate some other position-time graphs.

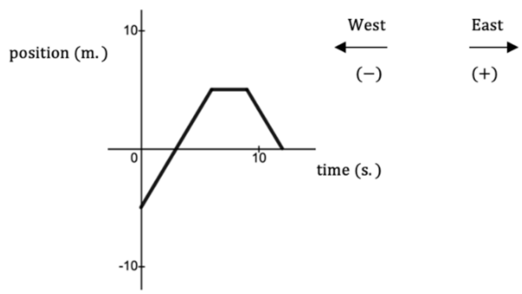
Looking Ahead 6.10



At 0 seconds, the object is 4 meters away $(0, 4)$ to the right (forward). At 2 seconds, the object is at the start $(2, 0)$. The object is not moving forward but backward to the start. At 4 seconds, the object is 4 meters away $(4, -4)$ to the left (backward). The object is moving backwards at a rate of $2\ \mathrm{m/s}$. The slope is $\frac{-4}{2} = -2$. The velocity is $-2\mathrm{m/s}$.

The object is moving backwards with uniform motion. This means that positive is forward, and negative is backward (by definition).

Example 1: An object is moving East if positive and West if negative. Explain from the position-time graph the motion when the object is at 0, 3, 6, 9, and 12 seconds.



Let us look at the slopes.

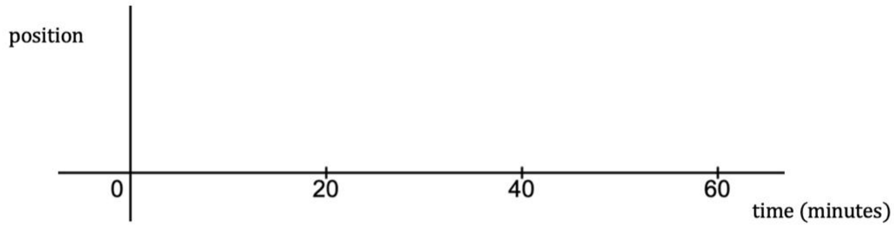
From 0 to 6 seconds the slope is $\frac{10}{6}$ $(m = \frac{5}{3})$. The object is moving at $1.67\ \mathrm{m/s}$. The slope is positive. The object is moving East.

From 6 to 9 seconds the line is horizontal. The slope is 0. That means $v = 0\ \mathrm{m/s}$. The velocity is 0 because the object is at rest.

From 9 to 12 seconds the slope is $-\frac{5}{3}$. The object is moving at a velocity of $1.67\ \mathrm{m/s}$ West because the slope is negative.

Example 2: Draw a graph of the following scenario:

You leave the house at a constant rate walking for 20 minutes. You come to a bench and rest for 20 minutes, then you walk back home over the next 20 minutes at the same constant rate.



A velocity graph is different than a position graph. Sometimes, distance-time graphs are position-time graphs. Sometimes, velocity graphs are called speed-time graphs. Read the labels. That is the only way you can tell the difference.

The speed of a moving object changes over time. A horizontal line on a velocity graph does not mean an object is not moving, it means that it is moving at a constant rate. The steeper the graph the greater the velocity. A less steep slope means an object is slowing down.

Example 3: Draw a velocity graph for the scenario in Example 2.

Section 6.11 Distance and Velocity are Related

Looking Back 6.11

In the previous Practice Problems section, the area under the velocity graph represented the total distance an object traveled. The slope of the distance-time graph was the velocity. Distance and velocity are related. In this section, we will investigate this relationship.

Looking Ahead 6.11

Example 1: Let us assume that you drive at a constant rate of 60 miles per hour. Let $v(t)$ be the velocity at a given time. Use this scenario to complete a)-d) below.

a) Draw the velocity graph for the interval $[0, 6]$ hours.

b) If acceleration is the rate of increase or decrease in velocity, what is your acceleration? Where do you see this on the graph?

c) What is $v(t)$ if $t = 2$?

d) What is $v(t)$ if $t = 6$?

Example 2: If you drive at 60 miles per hour, your distance increases at a steady rate. Use this information to complete a)-d) below.

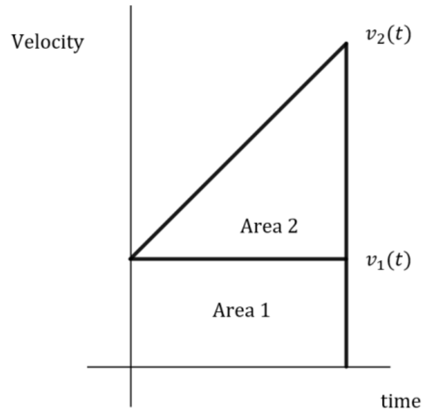
a) Draw the distance graph for the interval $[0, 6]$.

b) What is the distance function $d(t)$?

c) Find the slope of the tangent line to any point on the line of the graph of $d(t)$.

d) How does the slope relate to the velocity?

Below is a velocity graph to demonstrate how to find the area under a curve that is not a horizontal line.



The area under the curve is split into two areas. Area 1 is a rectangle (base $\cdot$ height) in which the base is the length of the time interval (t) and the height is velocity ($v_1(t)$).

Area 2 is a triangle with the same base as Area 1 but a height that is equal to $v_2(t) - v_1(t)$. This results in the change in y. If we take $\frac{1}{2}(t)(v_2(t) - v_1(t))$ we get the displacement. The area under the curve (a straight line in this case) is the total displacement caused by the motion depicted in the graph.

The area under the speed-time graph is the distance because speed and time are scalars as we learned at the beginning of this module. The area under the velocity-time graph is the displacement since velocity is a vector. Velocity is positive if the object is moving forward but may be negative if the object is moving backwards, as we have seen previously. Area under the curve is the magnitude of the displacement which is equal to the distance traveled for constant acceleration.

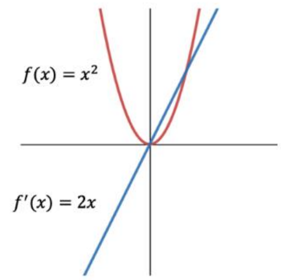
Section 6.12 Graphs of Functions and then Derivatives

Looking Back 6.12

In the previous section, we looked back at the sketches of graphs and compared position-time graphs to velocity graphs and distance-time graphs to velocity graphs. We learned that the slope of the distance-time graph is the velocity graph when we were comparing functions to their derivatives. In this section, we will further explore this topic.

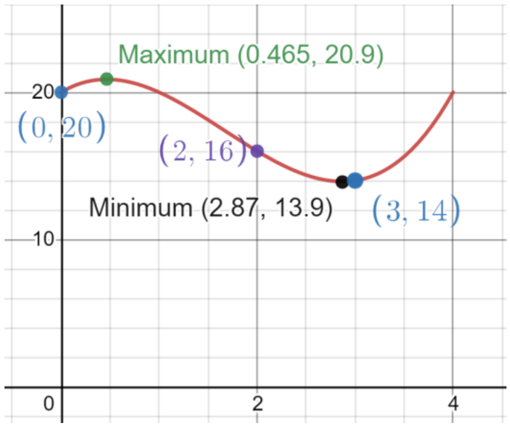
Looking Ahead 6.12

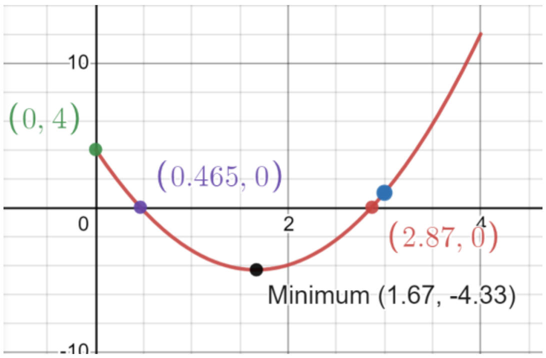
In the previous section, the velocity function $v(t) = 2t$ corresponded to the distance function $d(t) = t^2$. If we put the graphs together on one grid, we see that if $f(x) = x^2$, then $f'(x) = 2x$.



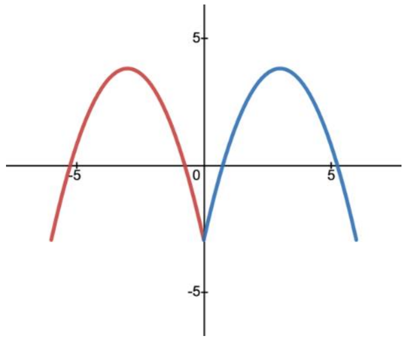
Example 1:	Compare the graph of the function $f(x) = x^2$ to its derivative $f'(x) = 2x$.

Example 2: Analyze position, displacement, velocity, distance, and speed of the motion of the yo-yo from 0 to 3 seconds.

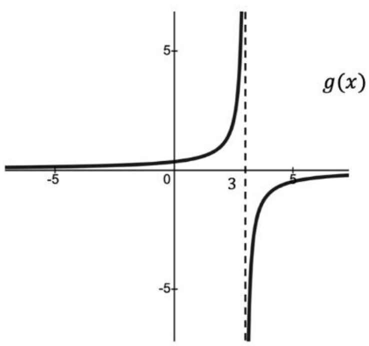




Example 3: Draw the graph of the derivative of the given function.



Example 4: Answer the questions that follow for the graph of $g(x)$ shown below and then draw the graph of its derivative $g'(x)$.



a) Is the slope of $g(x)$ increasing or decreasing from $-\infty < x < 3$? At what rate? Is it a positive or negative slope?

b) Is the slope of $g(x)$ increasing or decreasing from $3 < x < \infty$? At what rate? Is it a positive or negative slope?

c) From $-\infty < x < 3$ the slope is approaching a vertical line at the asymptote. Is this a vertical line or a horizontal line for $g'(x)$?

d) From $3 < x < \infty$ the slope is approaching 0. Is this a horizontal line or vertical line for $g'(x)$

Section 6.13 Distance, Velocity, and Acceleration

Looking Back 6.13

We have already defined speed as the time rate of change of the distance traveled by an object, and velocity as the time rate of change of an object's position. Hence, we have investigated what we call distance-time graphs and/or position-time graphs as well as velocity graphs, and speed-time graphs.

Speed and velocity may be defined as follows:

Speed:

$$\text{speed} = \frac{\Delta d}{\Delta t}$$

Δd is the change in distance of travel
Δt is the change in time it took to travel the corresponding distance

Velocity:

$$v = \frac{\Delta s}{\Delta t}$$

v represents velocity; s represents position (Δs is change in position); t represents time (Δt is change in time)

We have studied average velocity (the velocity of an object over an extended period of time), and instantaneous velocity (the velocity of an object at one moment in time) in this curriculum.

We will now discuss acceleration and how it relates to all that you have previously learned. It may be defined as "the time rate of change of an object's velocity."

Acceleration:

$$a = \frac{\Delta v}{\Delta t}$$

In this equation, a represents acceleration, Δv represents change in velocity, and Δt represents change in time.

Just as velocity measures how an object's position changes with time, acceleration measures how an object's velocity varies with time. And just as there is average and instantaneous velocity, there is also average and instantaneous acceleration. Therefore, if velocity is the first derivative of the position-time curve, then acceleration is the second derivative of the position-time curve.

An object is only accelerating if its velocity is changing. It can be changing its velocity by a constant amount. Constant acceleration is not the same as constant velocity. Acceleration takes two things into consideration, whether an object is speeding up or slowing down and whether an object is moving in a positive or negative direction.

Example 1: Describe the velocity and acceleration for an object that is traveling according to the given velocity-time table. Draw the velocity-time graph and acceleration-time graph given the table below.

Time (s.)	Velocity m./s.
0	0
1	2
2	4
3	6
4	8

Looking Ahead 6.13

Acceleration is an increase in velocity if an object is speeding up and a decrease in velocity if an object is slowing down. If velocity is measured in meters per second ($\mathrm{m/sec.}$), then acceleration is measured in meters per second squared ($\mathrm{m/sec.}^2$).

If an object has a velocity of $-5.3\ \mathrm{m/sec.}$ and an acceleration of $0.2\ \mathrm{m/sec.}^2$ or a velocity of $5.3\ \mathrm{m/sec.}$ and an acceleration of $-0.2\ \mathrm{m/sec.}^2$, the object is slowing down. In other words, if the signs of velocity and acceleration are opposite, then the object is slowing down. That makes sense: if an object's velocity and acceleration have opposite directions, the object will slow down.

If an object has a velocity of $-5.3\ \mathrm{m/sec.}$ and an acceleration of $-0.2\ \mathrm{m/sec}^2$, or a velocity of $5.3\ \mathrm{m/sec.}$ and an acceleration of $0.2\ \mathrm{m/sec.}^2$, then the object is speeding up. In other words, if the signs of the acceleration and the velocity are the same, the object is speeding up. That also makes sense: if an object's velocity and acceleration have the same direction, the acceleration is increasing the object's velocity. It is not decreasing it or slowing it down.

Example 2: Is the acceleration positive or negative for each of the situations in the following tables.

Time (s.)	Velocity m./s.
0	3
1	6
2	9
3	12
4	15

Time (s.)	Velocity m./s.
0	-3
1	-6
2	-9
3	-12
4	-15

Time (s.)	Velocity m./s.
0	15
1	12
2	9
3	6
4	3

Time (s.)	Velocity m./s.
0	-15
1	-12
2	-9
3	-6
4	-3

Example 3: A model rocket that is launched into the air has a velocity of $52.5\ \mathrm{m/sec.}$ at 5 seconds into flight and a velocity of $41.3\ \mathrm{m/sec.}$ at 7 seconds into flight. Use this information to answer the questions below.

a) What is the average acceleration of the model rocket over the interval $[5, 7]$?

b) What does the sign of acceleration tell us about the velocity of the model rocket? (We can see that the velocity is decreasing as it goes from $52.5\ \mathrm{m/sec.}$ to $41.3\ \mathrm{m/sec.}$ over 2 seconds.)

A good rule of thumb is that if an object is slowing down (as in the speed is decreasing), then the acceleration vector is in the opposite direction of the velocity vector. When speeding up, both vectors face the same direction.