Pre-Calculus and Calculus Module 8 Integrals and Integration

Section 8.1 Sequence and Series

Practice Problems 8.1

For Problem 1-10, tell whether the sequence given is an arithmetic sequence, geometric sequence, or neither.

1. $100, 50, 25 \ldots$
2. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$
3. $10, 16, 28 \ldots$
4. $3\sqrt{2}, 3, \frac{3\sqrt{2}}{2} \ldots$
5. $-13, -18, -23 \ldots$
6. $2, 20, 200 \ldots$
7. $4, 13, 40 \ldots$
8. $5, 24, 575 \ldots$
9. $x, x-4, x-8$
10. $n+2a, 2n+3a, 3n+4a \ldots$

For Problem 11-17, use the information given to solve the problem.

11. If an exponential function has the add and multiply property, which type of function has the multiply and add property?

12. If a linear function has an add and add property, which type of property does a power function have?

13. What is the fourth term of the sequence in Problem 6?

14. What is the fourth term of the sequence in Problem 10?

15. What is the explicit formula for the geometric sequence in Problem 1?

16. Use the “Seqgen” (Sequence Generator) command to find the first ten terms in Problem 5.

17. Use the “Seqn” (Explicit expression in terms of n, nMax) to find the first eight terms in Problem 1.

For Problem 18-20, use the given information to solve the problem.

You have been loved, cared for, and prayed for by your parents, grandparents, great grandparents, and beyond. Let us use sequences to see how many people have poured into your life!

18. Write the number of your biological relatives going back to the 5^{th} generation. What type of sequence is this?

19. What is the explicit formula for the number of your biological relatives going back to the 5^{th} generation? How many relatives do you have going back to the 15^{th} generation?

20. Continuing this process of counting relatives, the number will increase without limit and will exceed the world population. How is this so?

Section 8.2 Partial Sums and Series

Practice Problems 8.2

For Problem 1-5, use the given arithmetic sequence to solve the problem.

$$6, 10, 14, 18, 22, 26, 30, 34$$

1. Write the arithmetic sequence as an arithmetic series.

2. Find the partial sum for the first four terms.

3. Find S_6.

4. Use the partial sums formula to find the partial sum of the arithmetic series for the first eight terms.

5. What is the common difference?

For Problem 6-10, use the given geometric sequence to solve the problem.

$$200, 50, 12.5, 3.125, 0.78125$$

6. Write the geometric sequence as a geometric series.

7. Find the partial sum for the first four terms.

8. Find S_5.

9. Use the partial sums formula to find the partial sum of the geometric series for the first six terms.

10. What is the common ratio?

For Problem 11-14, use the information given to solve the problem.

11. Write the arithmetic sequence for Problem 1-5 using sigma notation and calculate the sum for the first six terms of the series. Is it the same sum as the sum found in Problem 3?

12. Does the series in Problem 6 converge or diverge as n approaches infinity for S_n?

13. Use the Binomial Theorem to expand the binomial $(2a - 2b)^3$.

14. Find the sixth term of the binomial expansion for the binomial $(2 + 3y)^{10}$.

For Problem 15-19, use the given geometric series to solve the problem.

A geometric series has a partial sum $S_n = \sum_{k=1}^{4} 3 \cdot (\frac{1}{2})^{k-1}$.

15. What is r (the geometric ratio)?

16. Find the sum of the series for the first four terms by expanding from the sigma notation.

17. Find an explicit formula for the nth partial sum.

18. Use the explicit formula from Problem 17 to find the partial sum for the first four terms of the series.

19. Determine whether the series converges as n approaches infinity. If the series converges, find its limit.

The Koch Curve Activity

Rather than completing practice problems, we will be performing another activity to investigate the Koch curve (Koch Island).

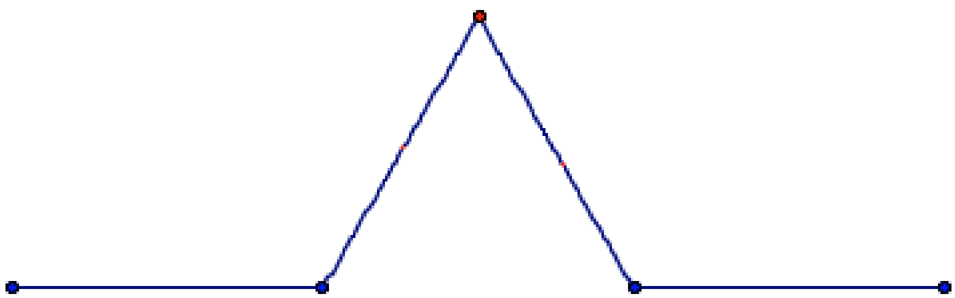
Here are the directions to make a Koch curve (you might remember this from Module 6 of Geometry and Trigonometry):

Step 1: Draw a straight line:

Step 2: Find and mark the midpoint that divides the line into two equal parts:

Step 3: Divide the whole line into three equal parts.

Step 4: Remove the middle piece and move the middle point up to form an equilateral triangle whose sides are as long as the missing piece:



Step 5: Keep repeating this process on each side until you form a Koch snowflake:

3. Complete the table below and answer the following questions:

Step	**Number of Equal Segments**	**Length of Each Segment (Units)**	**Total Length of Segment** **(Number of Equal Segments · Length of Each Segment)**
0	1	1	1
1	4	$\frac{1}{3}$	$4 \cdot \frac{1}{3} = \frac{4}{3}$
2	16	$\frac{1}{9}$	$16 \cdot \frac{1}{9} = \frac{16}{9}$
3	64	$\frac{1}{27}$	$64 \cdot \frac{1}{27} = \frac{64}{27}$
4			
n			

a) What is the ratio of any step “Total Length of Segments” to the previous step “Total Length of Segments?”

b) What is the constant multiplier?

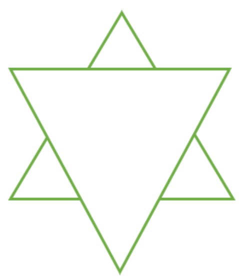
c) What would be the exponential function to represent the total length of segments at any step?

d) If the original segment at Step 0 were 27 units long instead of 1 unit long, how would that change the function?

e) Complete the table below with the original segment of 27 units:

Step Number	Total Length of Segments (Units)
0	27
1	
2	
3	

To look at the entire snowflake curve, begin with one equilateral triangle with lengths of sides three units, broken into one-unit lengths so equilateral triangles are formed on each side.



f) What is the perimeter of the first iteration shown above?

g) Draw the second iteration below. What is the perimeter?

h) This problem is an infinite ________________ series with a ratio of $r =$ ___________.

i) Will the perimeter converge or diverge?

Section 8.3 The Derivative and the Indefinite Integral

Practice Problems 8.3

For Problem 1-6, use the information from Example 1 in the Looking Back section to solve the problem.

1. Using Yolanda's velocity function, find $v(1)$, $v(2)$ and $v(t)$.

2. Using Yolanda's distance function, find $d'(1)$, $d'(2)$, and $d'(t)$.

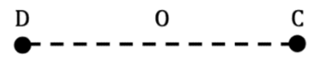
3. Find $d(2)$, the total distance traveled between $d(0)$ and $d(2)$?

4. Use the general formula $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ to find the tangent line of the distance function. First, rewrite the function by replacing f with d and x with t. What do you notice about your solution?

5. If you start with the velocity function, $v(t)$, how could you find the corresponding distance function?

6. If you start with the distance function, $d(t)$, how could you find the corresponding velocity function?

For Problem 7-20, use the diagram and scenario below to solve the problem.



A ball speeds up steadily as it moves from C to O, then slows down steadily from O to D. Then the ball speeds up steadily from D to O, and then slows down steadily from O to C. The rate of acceleration increases linearly with the distance of the ball point O.

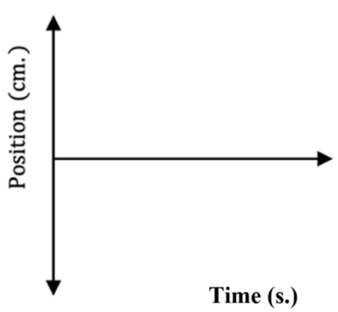
7. Describe the motion of the ball as it moves from C to D and back again.

8. Let O be the ball's equilibrium or rest, which is called the zero position. If there is positive motion from O to C, where is there negative motion?

9. If O is the center, when is the distance of the ball the greatest from the center? When is the distance of the ball the least from the center?

10. Where does the ball attain its greatest positive displacement from equilibrium? Where does is attain its negative displacement of greatest magnitude?

11. Draw a position verses time graph of the rolling magnetic ball from C to D and back to C. What trigonometric function models the curve of the graph?

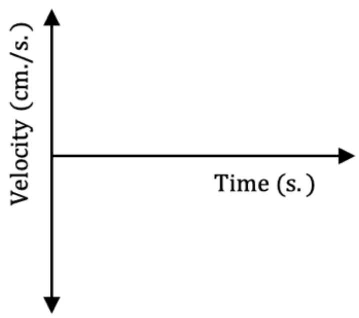


12. Velocity is speed with direction. If there is negative velocity from C to D, in which direction is there positive velocity?

13. When is velocity at a maximum?

14. When is velocity zero?

15. Draw a graph of one cycle (period) of the motion for the velocity of the rolling magnetic ball from C to D and back to C Assume the ball is in continuous motion.



16. What trigonometric function models the curve of the graph?

17. Does $\mathrm{OC} = \mathrm{OD}$ represent the amplitude or the frequency of the graph?

18. The equation for the velocity of the ball at any time (t in seconds) in which A is the amplitude is $y = \mathrm{A} \cdot \sin(\frac{2\pi}{p} \cdot t)$. What does p represent?

19. When the distance graph has a maximum amplitude, where is that point on the velocity graph?

20. When the velocity is the greatest, where is the position of the ball?

Section 8.4 Riemann Sums

Practice Problems 8.4

For Problem 1-4, write out the sums in expanded forma and calculate the sums.

1. $\sum_{j=1}^{4} (0.2j + 1.4)$

2. $\sum_{j=0}^{5} 2^j$

3. $\sum_{i=3}^{6} 4i - i$

4. $\sum_{m=4}^{9} m^3(\frac{1}{m})$

For Problem 5 and 6, use the information given to solve the problem.

5. Complete a)-g) for the summation of the expression $4 + 9 + 16 + 25 + 36 + 49 + 64 + 81$.

a) What is this famous series of numbers?

b) Rewrite the series as the sum of squares.

c) What are you adding to each base to get the next base?

d) Let $(k + 1)^2$ be the argument. What must the initial value of the index be to get 2^2?

e) What must the last value of the index be to get 9^2?

f) Write the sum using sigma notation.

g) Write out the summation and calculate its value. Does it match the sum of the given expression?

6. Write the sum of the following expression as a Reimann sum using sigma notation:

$$10 + 15 + 20 + 25 + 30 + 35 + 40 + 45$$

For Problem 7-12, find the partial sums of the function with the information given.

7. S_3 for $\sum_{n=1}^{\infty} 2n - 1$

8. S_1 for $\sum_{m=3}^{\infty} 4m + m^2$

9. S_2 for $\sum_{k=4}^{\infty} k + 6$

10. S_5 for $\sum_{t=1}^{\infty} t^3$

11. S_4 for $\sum_{v=1}^{\infty} 5 - v$

12. S_3 for $\sum_{x=0}^{\infty} \frac{x}{2}$

For Problem 13-16, use the given information to tell whether the series converges or diverges.

A geometric series is $\sum_{k=0}^{\infty} ar^k$ in which a is a constant and r is the common ratio. Remember, if $|r| < 1$, the series will converge, and if $|r| \geq 1$, the series will diverge.

13. $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} \ldots$

14. $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} \ldots$

15. $8 + 4 + 2 + 1 \ldots$

16. $5 + 15 + 45 + 135 \ldots$

For Problem 17-20, use the geometric series in Problem 14 to solve the problem.

17. What is "a" in the geometric series?

18. Use sigma notation to write a as an infinite series.

19. What is S_5?

20. If a geometric sequence converges it converges to $\frac{1}{1-r}$. What does this infinite series converge to?

Section 8.5 Left and Right Endpoint Rectangles for Approximating Area

Practice Problems 8.5

For Problem 1-4, name the index of the function given.

1. $\sum_{k=0}^{5} k(1+k)^2$

2. $\sum_{m=1}^{3} \frac{m^2}{16}$

3. $\sum_{z=1}^{4} z + z^2 + z^3$

4. $\sum_{p=0}^{10} \frac{p}{p^2-1}$

For Problem 5-8, find Δx using the interval and number of intervals given for the function f.

5. $[0,4]; n = 3$

6. $[0,10]; n = 5$

7. $[0,8]; n = 8$

8. $[0,12]; n = 36$

For Problem 9 and 10, write the sigma notation for the function $A(f, a \leq x \leq b)$ using the interval and the number of intervals given.

9. $[0,3]; n = 3$ using the left-endpoint method

10. $[2,6]; n = 4$ using the right-endpoint method

For Problem 11 and 12, approximate the area under the curve over the interval given using four left-endpoint rectangles.

11. $y = x^2 + 4;\ [-2,2]$

12. $y = \frac{x^2}{2} - x;\ [1,5]$

For Problem 13 and 14, use the information given to solve the problem.

13. What is the number of intervals for area under the curve if the interval is $[0, 5]$ and Δx is $\frac{1}{4}$?

14. What is the number of intervals for area under the curve if the interval is $[1, 3]$ and Δx is $\frac{1}{2}$?

For Problem 15 and 16, use the sum written in expanded form given to answer the questions/solve the problems that follow.

15.

$$\frac{3}{4}f\left(3-\frac{3}{4}\cdot 0\right)+\frac{1}{4}f\left(3-\frac{3}{4}\cdot 1\right)+\frac{1}{4}f\left(3-\frac{3}{4}\cdot 2\right)+\frac{1}{4}f(3-\frac{3}{4}\cdot 3)$$

a) What is Δx?

b) What is the lower bound of k? What is the upper bound of k?

c) How many intervals are there?

d) Write the sum as a Riemann sum using sigma notation.

16.

$$\frac{1}{2}f(-2+0.4\cdot 0)+\frac{1}{2}f(-2+0.4\cdot 1)+\frac{1}{2}f(-2+0.4\cdot 2)+\frac{1}{2}f(-2+0.4\cdot 3)$$

a) What is a, the start value of the interval?

b) What is x_0?

c) How many intervals are there?

d) What is b, the end value of the interval?

e) Write the sum as a Riemann sum using sigma notation.

For Problem 17, use the information given to solve the problem.

17. Draw the graph for the motion of a car that travels at $f(t) = 10t + 30$ miles per hour from 2 to 6 hours? Find the area under the trapezoid to find the distance from 2 to 6 hours by following the steps below.

18. The height of the trapezoid is the length of the interval on the $x - axis$. What number represents the height?

19. The $x - axis$ is perpendicular to the two bases. Find $b_1 = f(2)$ and $b_2 = f(6)$.

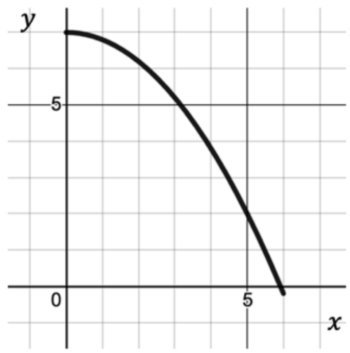
20. Use the formula for the area of a trapezoid, $A = \frac{1}{2}(b_1 + b_2)h$, to find the area under the curve which is the total distance from 2 to 6 hours.

Section 8.6 Midpoint and Trapezoidal Methods

Practice Problems 8.6

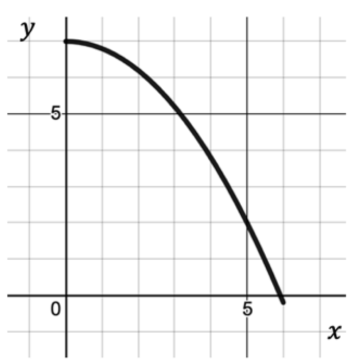
For Problem 1-18, use the information given to solve the problem.

1. Find the area under the curve of the function $-\frac{1}{5}x^2 + 7$ using left-endpoint rectangles from 2 to 5 and three sub-intervals.



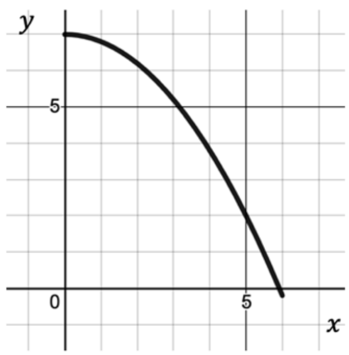
2. For the function $f(x) = \frac{1}{10}x^2 + 3$, the left-endpoint rectangles resulted in an underestimate of area under the curve. For the function in Problem 1, will using the left-endpoint rectangles result in an underestimate or overestimate of area? Explain why.

3. Find the area under the curve $-\frac{1}{5}x^2 + 7$ using the right-endpoint rectangles from 2 to 5 and three sub-intervals.



4. Is the area in Problem 3 an overestimate or underestimate of area under the curve of the function over the given interval? Explain why.

5. Find the area under the curve $f(x) = -\frac{1}{5}x^2 + 7$ using midpoint rectangles and three sub-intervals over the interval from 2 to 5.



6. Compare the three areas from Problem 1, 3, and 5. What is similar in each solution? What is different in each process? Which is the best approximation and why?

7. Which would give a better estimate of area for Problem 5, three sub-intervals or ten sub-intervals?

8. Which Δx is larger for Problem 5, three sub-intervals or ten sub-intervals?

9. Using the function $f(x) = -\frac{1}{5}x^2 + 7$, how big is Δx for ten sub-intervals over the interval 2 to 5?

10. Find $f(1.5)$ for $f(x) = -\frac{1}{5}x^2 + 7$.

11. Find $f(0.5)$ for $f(x) = -\frac{1}{5}x^2 + 7$.

12. Why is $f(0.5) > f(1.5)$?

13. What is the approximate area from $[0.5, 1.5]$ for the function $f(x) = -\frac{1}{5}x^2 + 7$ using the midpoint method?

14. Write an inequality for the approximate area under the curve from Problem 10-13.

15. Using sigma notation for A($-\frac{1}{5}x^2 + 7, 2 \leq x \leq 5$), what is a?

16. Using sigma notation for A($-\frac{1}{5}x^2 + 7, 2 \leq x \leq 5$) and three sub-intervals, what is Δx?

17. Write the function using sigma notation:

$$\Delta x \int_{k=0}^{n-1} f(a + \Delta x\left(\frac{1}{2} + k\right))$$

18. Using the Riemann sum from Problem 17, approximate the area under the curve for the function $A(-\frac{1}{5}x^2 + 7, 2 \leq x \leq 5)$. Is it the same as the area in Problem 5?

For Problem 19 and 20, use the trapezoidal method and four sub-intervals to approximate the area under the curve for the function and interval given.

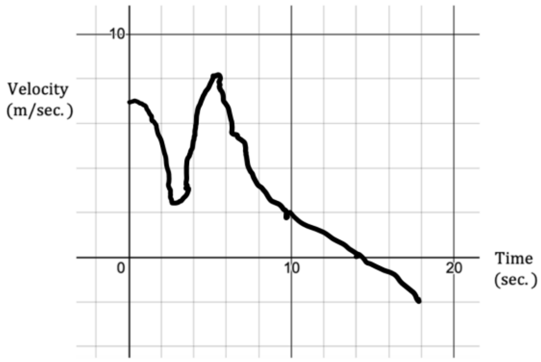
19. $y = -\frac{x}{2} - x + 5;\ [2, 6]$

20. $y = \frac{4}{x};\ [4, 5]$

Section 8.7 Negative Area Under the Curve

Practice Problems 8.7

For Problem 1-8, use the given velocity versus time graph of a runner's motion to solve the problem.



1. When does the runner change directions?

2. When is the runner moving the fastest?

3. When is the runner running at a speed of 2 m./sec.?

4. When is the runner speeding up?

5. How do you know when the runner slowing down?

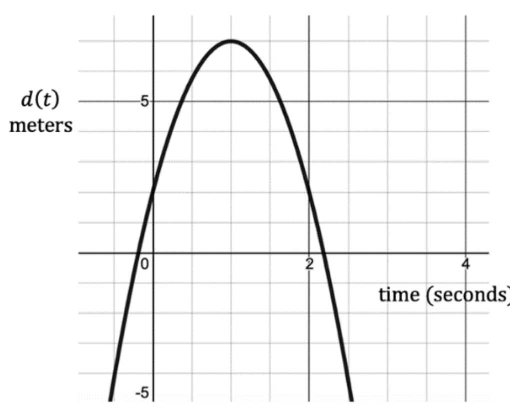
6. At what interval of time(s) is the runner slowing down?

7. When does the runner momentarily stop?

8. When does the runner turn around and start running in the direction he came from?

For Problem 9-13, use the given information and graph to solve the problem.

A ball is thrown into the air such that its height above the ground h as a function of time t is modeled by the function $h(t) = -5t^2 + 10t + 2$ in which the height is measured in meters.



9. Approximately when is the ball at its maximum height?

10. At what time does the ball change direction from ascending to descending?

11. At what time does the ball hit the ground?

12. Given $h(t) = -5t^2 + 10t + 2$, find the velocity function of the ball and graph it. Let the distance above the ground d be the height h.

13. Find the area under the velocity curve between 0 and 2 seconds. What does it represent? What does the sum of the absolute value of each of the areas represent?

For Problem 14-17, use the given information and table to solve the problem.

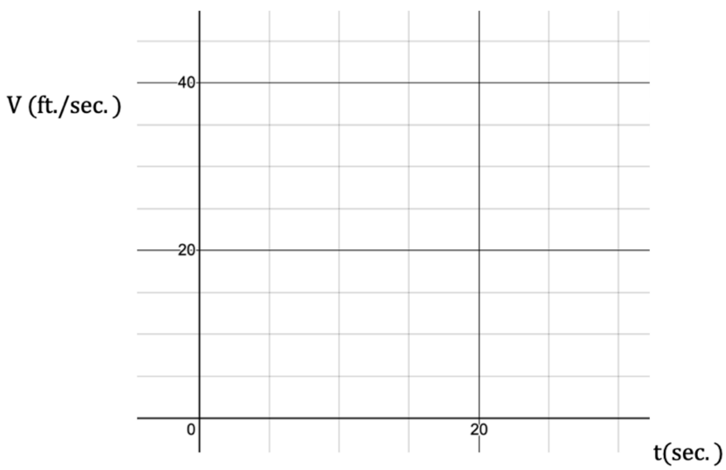
A car's odometer is broken so we want to use the speedometer readings, taken every five seconds, to estimate the distance traveled over a 30-second interval.

Time (**sec**.)	0	5	10	15	20	25	30
Velocity (**mi**./**hr**.)	16	20	24	30	32	32	29

14. Convert the velocity to feet per second and complete the table below ($1\ \frac{\text{mi.}}{\text{hr.}} = \frac{5{,}280\text{ ft.}}{3{,}600\text{ sec.}}$). Round to the nearest whole number.

Time (**sec**.)	0	5	10	15	20	25	30
Velocity (**ft**./**sec**.)							

15. Graph the points from the table and connect them to make a curve. Use these as left-endpoints and draw the rectangle of the area under the curve.



16. Find the area under the curve (Note that the area consists of rectangles and trapezoid).

17. What does the sum of the areas represent?

For Problem 18-20, using the given information, match the expression given to its correct description. Suppose the function $v(t)$ gives the velocity (v) of an object as a function of time (t). (Assume the object moves one-dimensionally). Consider the expressions below that may be used to determine the displacement of an object over the interval $t = [a, b]$.

18. Riemann sum approximation (left endpoint)

19. Trapezoid method approximation

20. Riemann sum approximation (right endpoint)

a) $d = \frac{1}{2}v(a)\Delta t + v(a+\Delta t)\Delta t + v(a+2\Delta t)\Delta t + \cdots$

$\ldots + v(b-2\Delta t)\Delta t + v(b-\Delta t)\Delta t + \frac{1}{2}v(b)\Delta t$

$$d = \Delta t\left[\frac{v(a)+v(b)}{2} + \sum_{k=1}^{n-1} v(a+k\Delta t)\right]$$

b) $d = v(a+\Delta t)\Delta t + v(a+2\Delta t)\Delta t + \cdots$

$\ldots + v(b-2\Delta t)\Delta t + v(b-\Delta t)\Delta t + v(b)\Delta t$

$$d = \Delta t \sum_{k=1}^{n} v(a+k\Delta t)$$

c) $d = v(a)\Delta t + v(a+\Delta t)\Delta t + v(a+2\Delta t)\Delta t + \cdots$

$\ldots + v(b-2\Delta t)\Delta t + v(b-\Delta t)\Delta t$

$$d = \Delta t \sum_{k=0}^{n-1} v(a+k\Delta t)$$

Section 8.8 Using Technology to Calculate Riemann Sums

Practice Problems 8.8

For Problem 1, use the information given to solve the problem.

1. The second calculation of Riemann sums was in Example 2 of Section 8.2. Check it to see if $\sum_{n=1}^{3} (\frac{n}{n+1})$ is approximately equal to 1.91667. Make sure to type it as follows:

$$\sum_{n=1}^{3} (\frac{n}{(n+1)})$$

For Problem 2-5, use your calculator to check the answers you found by hand in Problem 1-4 of Section 8.2.

2. $\sum_{j=1}^{4} (0.2j + 1.4)$

3. $\sum_{j=0}^{5} 2^j$

4. $\sum_{i=3}^{6} 4i - i$

5. $\sum_{m=4}^{9} m^3(\frac{1}{m})$

For Problem 6-9, use your graphing calculator to find the partial sums.

6. S_3 for $\sum_{t=1}^{3} 2t - 1$

7. S_5 for $\sum_{m=1}^{5} m^3$

8. S_4 for $\sum_{v=1}^{4} 5 - v$

9. S_3 for $\sum_{k=0}^{3} \frac{k}{2}$

For Problem 10, use the given information to solve the problem.

We used all four methods to find the area under the curve for the motion function $\frac{1}{10}x^2 + 3$. The area using the midpoint rectangle method was found to be 19.125.

The derived sigma notation was found to be:

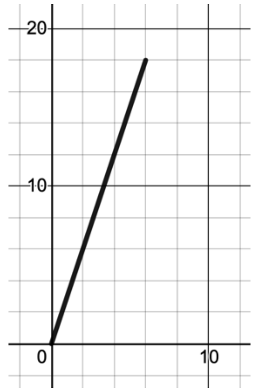
$$\sum_{k=0}^{4} \frac{1}{10}(0.5 + k)^2 + 3$$

Use your calculator and this sigma notation check the partial sum.

Section 8.9 Antiderivatives and the Definite Integral

Practice Problems 8.9

For Problem 1-8, use the given graph of $f(x) = 3x$ to solve the problem.



1. What is the lower bound of the interval?

2. What is the upper bound of the interval?

3. What is the function of the integrand?

4. Use a triangle to find the areas under the curve over the interval $[0, 6]$ for the function $f(x) = 3x$.

5. Given the antiderivative, fill in blank:

$$\int_0^6 3x dx = \frac{3}{2}x^2 \Big|_0^6$$
$$= \frac{3}{2}(6)^2 - \frac{3}{2}(0)^2$$
$$= \frac{3}{2}(36) - 0$$
$$= 54$$

Finding the antiderivative of a polynomial is the same as using reverse thinking for the ____________________ Rule.

6. What is the derivative of $\frac{3}{2}x^2$?

7. Using technology to find the region representing $\int_6^0 3x\,dx$ results in -54. How is this different than $\int_0^6 3x\,dx$?

8. The region bounded by $\int_6^0 3x\,dx$ is the same as the region bounded by $\int_0^6 3x\,dx$. However, for $\int_0^6 3x\,dx$, the area is positive and for $\int_6^0 3x\,dx$, the area is negative. Does this mean $\int_a^b f(x)dx$ is equal to $\int_b^a f(x)dx$ or $\int_a^b f(x)dx$ is equal to $-\int_b^a f(x)dx$?

For Problem 9-17, use the information given to solve the problem.

9. If a function is $8x$, then the general antiderivative is $4x^2 + \mathrm{C}$. If a function is $18x$, what is the general antiderivative?

10. If a function is $-10x$, what is the general antiderivative?

11. If the derivative of $\sin x$ is $\cos x$, what is the general antiderivative for $\cos x$?

12. If the derivative of the $\cos x$ is $-\sin x$, what is the general antiderivative of $\sin x$?

13. What is the general antiderivative of the function $\cos 3x$?

14. What is the general antiderivative of the function $\sin 2x$?

15. What is the general antiderivative of the function $4x^3$?

16. What is the derivative of x^4?

17. If the general antiderivative of the function $\frac{du}{dx} + \frac{dv}{dx}$ is $u + v + c$ using the Sum Rule, what is the general antiderivative of $\frac{du}{dx} - \frac{dv}{dx}$ using the Difference Rule?

For Problem 18-20, tell whether the statement is true or false.

18. If F' is equal to 0, then F is a constant.

19. If functions with the same derivative differ only by a constant C such that $F(x)$ is equal to $\mathrm{G}(x) - \mathrm{C}$, then $\mathrm{F}'(x)$ is equal to $\mathrm{G}'(x)$ for all values x.

20. Let function F be the antiderivative of the function f over any given interval I. Then $\mathrm{F}'(x)$ is equal to $f(x)$ at every point of I except points of discontinuity.

Section 8.10 The Fundamental Theorem of Calculus

Practice Problems 8.10

Today, we will be making a Streusel Cake in a coffee mug. It takes about 5 minutes to make and only makes enough for 1 person (unless you only eat half and decide to share it).

The ingredients and directions are listed below the problems. The answers to the Practice Problems will tell us the amount of each ingredient needed to make the cake. We can check our answers in the solution video or by tasting the cake!

For Problem 1-6, use the information given to solve the problem.

1. The velocity of an object traveling in meters per second is represented by the function $v(t) = t^2 + 2t + 1$. What is the velocity of the object at 0.25 seconds? (Only include values to the tenth place with no rounding.)

2. What is the acceleration of the object at $t = 0.5$?

3. What is the net displacement of the object from $t = 0.25$ seconds to $t = 0.4$ seconds? If we convert it to a fraction, is it closest to $\frac{1}{2}$, $\frac{1}{3}$, or $\frac{1}{4}$?

4. Evaluate $\int_0^3 \frac{1}{24}(4 - x^2)dx$ as $\frac{1}{24}\left[\int_0^3 4dx - \int_0^3 x^2dx\right]$.

5. Evaluate $\int_{\pi}^{2\pi} \cos x \, dx$.

6. Evaluate the limit: $\lim_{x \to 3} \frac{x^2-5x+6}{x-3}$.

The actual ingredients for the Streusel Cake in a coffee mug and the amounts are to be written below from your solutions Problem 1-6.

Ingredients for Cake

Answers to:

#2 tbsp. baking flour

#4 tsp. salt

#4 tsp. baking powder

#6 tbsp. sugar

#1 tbsp. buttermilk (add 2 drops of lemon juice to the milk to make it buttermilk)

#6 tbsp. melted butter

#3 tsp. vanilla

Ingredients for Topping

#5 tbsp. chopped pecans or walnuts

#6 tbsp. brown sugar

#6 tsp. corn oil

#3 tsp. cinnamon

#4 tsp. salt

Ingredients for Glaze

#3 cup(s) of powdered sugar

#6 tbsp. milk

Directions for Streusel Cake in a Mug

1. To make the cake: Combine the flour, baking powder, salt, and sugar in a greased coffee mug (you can use the wrapper from a stick of butter to grease it). Mix it well. Then add the buttermilk, melted butter, and vanilla, and mix well until it is all combined.

2. To make the topping: In another small bowl, mix the walnuts or pecans, brown sugar, corn oil, cinnamon, and salt. Mix it well. Heap spoonfuls of the topping on top of the batter in the mug for 10-second intervals in the microwave up to 1-minute. You can insert a toothpick into the mix to figure out when it is done. You will know because the toothpick will come out clean. If it comes out with batter on it, keep heating it up.

3. To make the glaze: In another small bowl, combine the powdered sugar and buttermilk and stir until it is smooth and runny. Drizzle it over the cake when it is done. Enjoy!

Section 8.11 Antiderivatives and the Indefinite Integral

Practice Problems 8.11

For Problem 1-6, tell which type of integral answers the question: definite integral or indefinite integral.

1. Which type of integral would you use to find the area under $g(x) = -3x^2 - 4x + 2$ over the interval $[-1, 5]$?

2. Which type of integral gives a numerical answer?

3. Which type of integral gives an answer in the form of a function?

4. Which type of integral finds all the slope functions for $h(x) = -2\cos 3x$?

5. Which type of integral is unbounded?

6. Which type of integral has a defined boundary?

For Problem 7-10, evaluate the indefinite integral and find the antiderivative using the reverse power rule.

7. $\int 5dx$

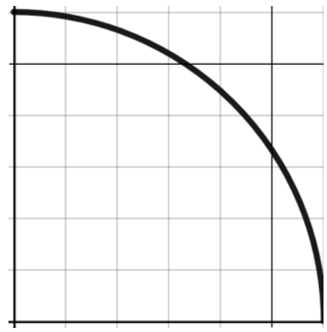
8. $\int \left(\frac{1}{2} - m\right) dm$

9. $\int (-3x + 7)dx$

10. $\int (x^3 - 4x^2 - 6)dx$

For Problem 11-20, complete each activity/problem to find the average (mean) value of a function. All of the functions drawn are perfect quarter circles.

11. On the 6x6 grid below, highlight any square that is completely within the curve in yellow. Highlight any square that contains part of the curve or none of the curve in blue.



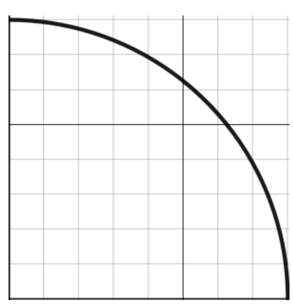
12. Complete the table below for the 6x6 grid from Problem 11.

Column	1	2	3	4	5	6
Squares Highlighted In Yellow						

13. What is the average number of yellow highlighted squares in a column? Using this, calculate the average value of the function.

14. Use $A = \pi r^2$ and remember this area is one-fourth of the full circle. What is the exact answer for the average value of the function?

15. On the 8x8 grid below, highlight any square that is completely within the curve in yellow. Highlight any square that contains part of the curve or none of the curve in blue.



16. Complete the table below for the 8x8 grid.

Column	1	2	3	4	5	6	7	8
Squares Highlighted In Yellow								

17. What is the average number of yellow highlighted squares in a column? What is the average value of the function?

18. What is the exact answer for the average value of the function?

19. As the grids includes more squares, what happens to the average value of the function?

20. The area under the curve is represented by the integral $\int_a^b f(x)dx$. Write an expression for the average value of a function.

Section 8.12 Rules of Integration

Practice Problems 8.12

For Problems 1-10, use the information given to solve the problem.

1. Evaluate $\int_{\frac{\pi}{2}}^{0} \cos x \, dx$. How is this different than $\int_{0}^{\frac{\pi}{2}} \cos x \, dx$? How does this change the solution?

2. If $b < a$, what does that tell you about Δx? If $b > a$, what does that tell you about Δx?

3. Just as the natural exponential function is its own derivative, so too the natural exponential function is its own integral plus the constant c:

$$\int e^x dx = e^x + c$$

Fill in the blanks to evaluate the integral:

$$\int e^{2x} dx$$

What is the differential of the inside function (the exponent)? __________dx

What constant do we have to multiply that number by to make sure that we have a clever form of 1?

____

__________ $\int e^{2x}$(________)dx

__________ $e^{2x} + c$

4. Evaluate $\int 3^x dx$.

Fill in the blanks to evaluate the indefinite integral:

$$\int 3^x dx$$

$$= \_\_\_\_\_\_\_\_ \int 3^x \ln 3 \, dx$$

$$= \_\_\_\_\_\_\_\_ \cdot 3^x + c$$

$$= \frac{3^x}{\_\_\_\_\_\_\_\_} + c$$

5. Evaluate $\int e^{ln3x} dx$.

6. Evaluate $\int_1^3 [(x^2+1)-(5x)]dx$. Rewrite it as two integrals first.

7. Evaluate $\int_{-2}^{4} -6(x)^3 dx$. Rewrite it as $c\int_{-2}^{4} f(x)dx$ first.

8. Evaluate $\int_{-1}^{1} 4x^3 dx + \int_1^3 4x^3 dx$. Rewrite it as one integral first.

9. Write the sum rule for the indefinite integral $\int [f(x) + g(x)]dx$.

10. Evaluate $\int_{1}^{3} [(x^3 - 2) + (2x^2)]dx$.

For Problem 11-14, use the given information to solve the problem.

Use the sum rule for definite integrals. Let $\int_{1}^{5} f(x)dx$ be -2, $\int_{3}^{5} g(x)dx$ be 7, and $\int_{3}^{5} h(x)dx$ be 4 such that f, g, and h are continuous.

11. $\int_{1}^{5} 6f(x)dx$

12. $\int_{3}^{5} [g(x) + h(x)]dx$

13. $\int_{5}^{1} f(x)dx$

14. $\int_{5}^{1} -4f(x)dx$

For Problem 15-17, let g be continuous and let $\int_2^3 g(x)dx$ be -5 and $\int_3^5 g(t)dt$ be 4 to solve the problem.

15. $\int_2^2 g(x)dx$

16. $\int_1^4 g(s)ds$

17. $\int_2^5 g(y)dy$

For Problem 18-20, use the information given to solve the problem.

18. Find the average value of $f(x) = 3x + 2$.

19. Evaluate $\int_0^1 (4x^3 - 2x + 1)dx$.

20. Evaluate $\int_0^4 \frac{2}{x+1}dx$.

Section 8.13 Using Technology to Evaluate Integrals

Practice Problems 8.13

For Problem 1-10, follow the instructions to evaluate the integral by hand and check your solution using a calculator.

1. What is $\int_{0}^{2} (5x^4 - 4x^3 + 3x^2 - 2x)dx$?

2. Find the average or mean value of $f(x) = 3x + 5$ over the interval $[1, 6]$.

3. Because $\frac{d}{dx} e^x$ is equal to e^x, what is $\int e^x dx$? (Add the + C in the solution).

4. The derivative of e^x is e^x. The derivative of e^{-x} is $-e^{-x}$. Calculate $\int_{-1}^{2} -3e^{-x} dx$.

5. The antiderivative of an exponent is the natural logarithm: $\int x^{-1} dx = \ln x$ or $\int \frac{1}{x} dx = \ln x$. Calculate $\int_{0}^{4} \frac{2}{x+1} dx$.

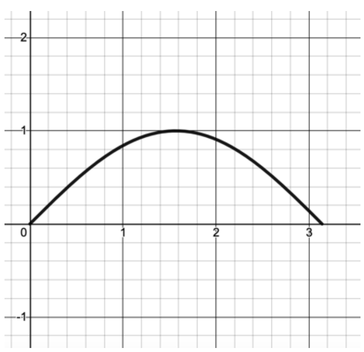
6. What is $\int_{1}^{y} \frac{1}{t} dt$?

7. Because $\int_a^b c\,dx$ is equal to $c(b-a)$, find $\int_1^4 7\,dx$.

8. What is $\int_1^4 7\,dt$?

9. What is $\int_0^{\pi+2} \sin(x-2)\,dx$?

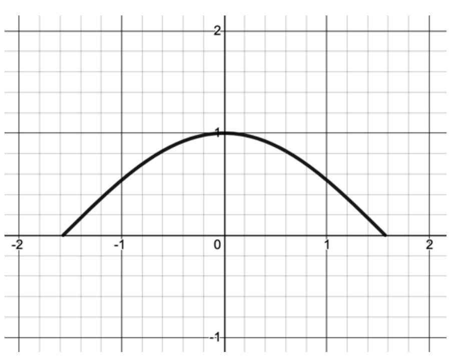
10. a) Find the area under one arch of the sine curve.



Step 1: Find the antiderivative, $F(x)$, of $f(x) = \sin x$.

Step 2: Calculate $F(\pi) - F(0)$ or $\int_0^{\pi} \sin x\,dx$.

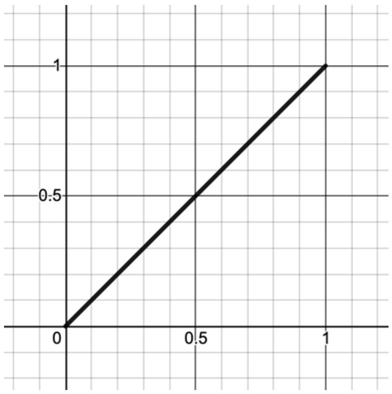
b) Find the area under one arch of the cosine curve.



Step 1: Find the antiderivative, $F(x)$, of $f(x) = \cos x$.

Step 2: Calculate $F\left(\frac{\pi}{2}\right) - F(-\frac{\pi}{2})$ or $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\ dx$.

c) What is the area under the curve for $y_1 = x^1$ in the interval $[0, 1]$?



Step 1: What is the area under the curve for $y_2 = x^2$ in the interval $[0, 1]$?

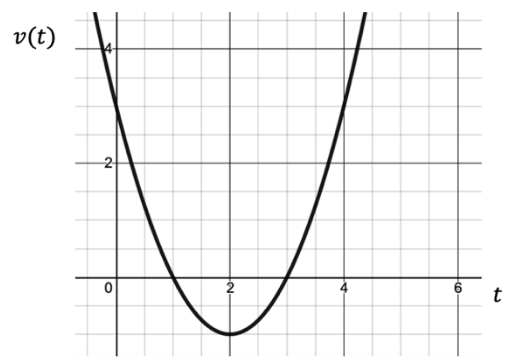
Step 2: What is the area under the curve for $y_3 = x^3$ over the interval $[0, 1]$?

Step 3: Make a conjecture for the explicit value of the area under the curve for y_n.

For Problem 11-14, use the given information to solve the problem.

The velocity of a moving object in furlongs per minute as a function of time (t) in minutes is given by the following equation:

$$v(t) = (t-1)(t-3)$$
$$v(t) = t^2 - 4t + 3$$



11. What is the velocity of the moving object at $t = 2$?

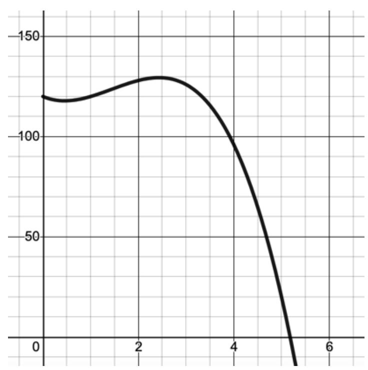
12. What is the acceleration of the moving object at $t = 2$?

13. What are the velocity and acceleration of the moving object at $t = 2.5$?

14. Is the object speeding up or slowing down at $t = 2.5$? Explain your reasoning.

For Problem 15-17, use the given information to solve the problem.

The viscosity of normal motor oil is its resistance to flow measured in centipoise. It decreases as temperature increases. Suppose that 10w-30 motor oil has a viscosity of $\mu = 120 - 10\text{T} + 13\text{T}^2 - 3\text{T}^3$. The Greek letter "$mu$" ($\mu$) is the viscosity in centipoise, the temperature is T in degrees Celsius. This equation is for temperatures from $0°$ to $200°$. This is a problem worth solving but some parts may not yield solutions which are possible.



15. Find the temperature in the domain when the maximum viscosity occurs. Set $\frac{d\mu}{dt}$ equal to 0 and solve for T. (Note- You will see that this solution gives a viscosity below ~5° which is negative and very unrealistic).

16. Find the minimum viscosity over the interval $[0, 200]$. This leads to another impossible value, but I hope you are having fun finding a solution to these highly unlikely problems!

17. The oil heats up such that T is equal to $\sqrt{t}$ in which t is time in minutes. What is the rate of change for viscosity when the temperature is $100°$? This requires observation at $t = 10{,}000\ min.$ or nearly seven hours!

For Problem 18-20, tell whether the statement is true or false.

18. The derivative $\frac{dy}{dt}$ of a function $y(t)$ is the slope of the tangent line to that function at time t.

19. Displacement is always positive.

20. $\int a dx = \int \left(v \cdot \frac{dv}{dx}\right) dx = \int (v) dv = 2v^2$

Section 8.14 Derivatives and Integrals

Practice Problems 8.14

For Problem 1-7, find the derivative of the function given.

1. $\frac{d}{dx}\ \ln(3x-7)^4$

2. $\frac{d}{dx}(e^{-x})$

3. $\frac{d}{dx}(e^{2x})$

4. $\frac{d}{dx}\ln(2x^2-x)$

5. $\frac{d}{dx}\ln(3x+5)$

6. $\frac{d}{dx}\left[\int\ \tan(x)\right]dx$

7. $\frac{d}{dx}\left[\int 2\sin 2x\right]$

For Problem 8, use the information given to solve the problem.

8. Differentiate $y=(x^4+3)$ with respect to x.

For Problem 9-14, evaluate the indefinite integral given.

9. $\int x\ \cos x\ dx$

10. $\int 2\sin 2x$

11. $\int (4x-1)^3 dx$

12. $\int\ \sin 3x\, dx$

13. $\int (3x+2)^2 dx$

14. $\int -20x^3 dx$

For Problem 15, fill in the blanks to evaluate $\int x^2 \cos 2x \, dx$.

15.

Let $u = x^2$, $dv = \cos 2x \, dx$, $du = 2xdx$, and $v = \frac{1}{2}\sin 2x$.

Use the indefinite $\int udv = uv - \int vdu$ to fill in the blanks that make the statement true.

$$x^2 \cdot \_\_\_\_\_\_\_\_\_\_ - \int 2x\left(\frac{1}{2}\sin 2x\right)dx$$

The integral on the right side is still a product of two functions. Use integration by parts to fill in the blanks that follow:

Let $u = 2x$, $dv = \frac{1}{2}\sin 2x$, $du = 2dx$, and $v = -\frac{1}{4}\cos 2x$.

$$\frac{1}{2}x^2 \sin 2x - \left[(\_\_\_\_\_\_\_\_)(-\frac{1}{4}\cos 2x) - \int (\_\_\_\_\_\_\_\_\_\_ \; 2dx\right]$$

$$\frac{1}{2}x^2 \sin 2x - \left[(\_\_\_\_\_\_\_\_\_\_) + (\frac{1}{4}\sin 2x)\frac{1}{2}\right]$$

$$\frac{1}{2}x^2 \sin 2x - \frac{1}{2}x\cos 2x + \_\_\_\_\_\_\_\_\_\_ + c$$

For Problem 16-18, find the definite integral of the function given.

16. $\int_0^{\frac{\pi}{2}} x \sin x \; dx$

17. $\int_{\pi}^{\frac{3\pi}{2}} \cos 5x \, dx$

18. $\int_1^{20} \frac{1}{x+5} dx$

For Problem 19-20, use implicit differentiation to find $\frac{dy}{dx}$ for the equation given.

19. $2x + 3y + 4y^2 = 7$

20. $3x^2 + 4x^3y = -y^3 + 5$ (Hint: use the product rule for $4x^3 \cdot y$; let $u = y$ and $v = 4x^3$.)

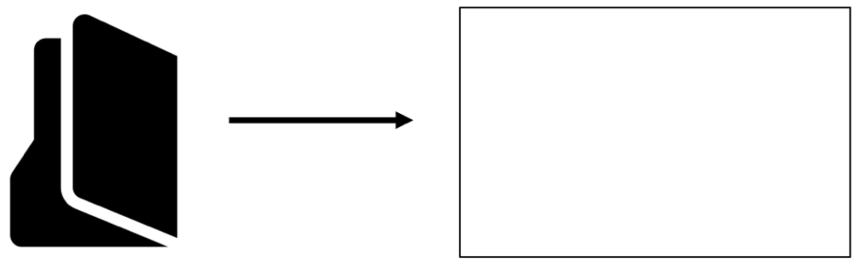
Section 8.15 Solids of Revolution

Practice Problems 8.15

Instead of completing practice problems, in this section we will be building a model to display the volume of a region. Cardboard partitions will be used to approximate the volume of a solid.

First, we will construct the solid:

1. Open a file folder and glue graph paper to the right and left side of it. Draw an x-axis on the right side about an inch from the bottom of the folder from 0-4 inches with 16 sub-intervals. Let one sub-interval be one unit on the graph paper. Draw a y-axis in the middle of the folder along the left edge of the graph paper. Mark this 1-16.



2. Draw an accurate sketch of the region bounded above by $y = 2^x$ and bounded below by $y = 0$, bounded to the left by $x = 0$ and bounded to the right by $x = 4$.

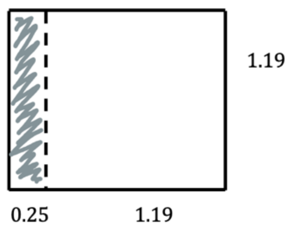
3. Shade in this region, which will be the base of the 3-dimensional solid.

4. Divide the 4 inches marked along the x-axis into 16 sub-intervals. Find Δx. At each endpoint of each sub-interval make a parallel cross-section of a solid revolved over the axis so that each section is perpendicular to the base of the solid and the x-axis. Each cross-section will be the shape of a square whose width is 2^{x_k} and length is 2^{x_k}. These will actually be the length and the height of the 3-dimensional approximated volume of the solid. Add Δx or any length less than Δx to the length to be scored and folded.

Glue this portion to the folder at each sub-interval to create a 3-dimensional model of the solid.

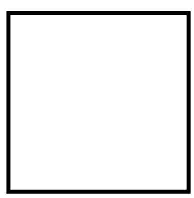
Example 1: At x_1, the value of x is equal to 0.25 and $f(x_1)$ is equal to 1.19.

$$y = 2^{0.25} = 1.19$$



Example 2: At x_{16}, the value of x is equal to 4, and $f(x_{16})$ is equal to 16.

$$y = 2^4 = 16$$



Use a colored folder to cut out the cross-sections. The y-axis is not accurately scaled to inches due to the size of the folder so just move pieces of cardboard from the colored folders next to the sketch of the graph of $y = 2^x$ at x_1 to mark the height of x_1 at that point on the cardboard.

Make the other three sides of the square the same length of $f(x_1)$. Add Δx to one side of the folder and glue the base of the solid so the square is placed upright perpendicular to both the base and the x-axis.

Approximate the volume of the model:

5. We will be finding the volume of each partition. Assume each partition is 3-dimensional. We can put pieces of playdough between each partition to get a visual. It is not as easy to build as with cardboard, though.

6. First, use the left-endpoint and then the right-endpoint method to approximate the volume of the shapes under the curve. Find the area of each partition and then multiply by the height and add them all together.

$$\mathrm{V} = \mathrm{B} \cdot h \text{ and } \mathrm{B} = l \cdot w$$

$$\text{Area} = f(x_k)dx \text{ or } f(x_k)\Delta x \text{ (at each endpoint)}$$

$$\text{Height} = f(x_k) \text{ so Volume} = f(x_k)f(x_k)dx \text{ or } f(x_k)^2\Delta x$$

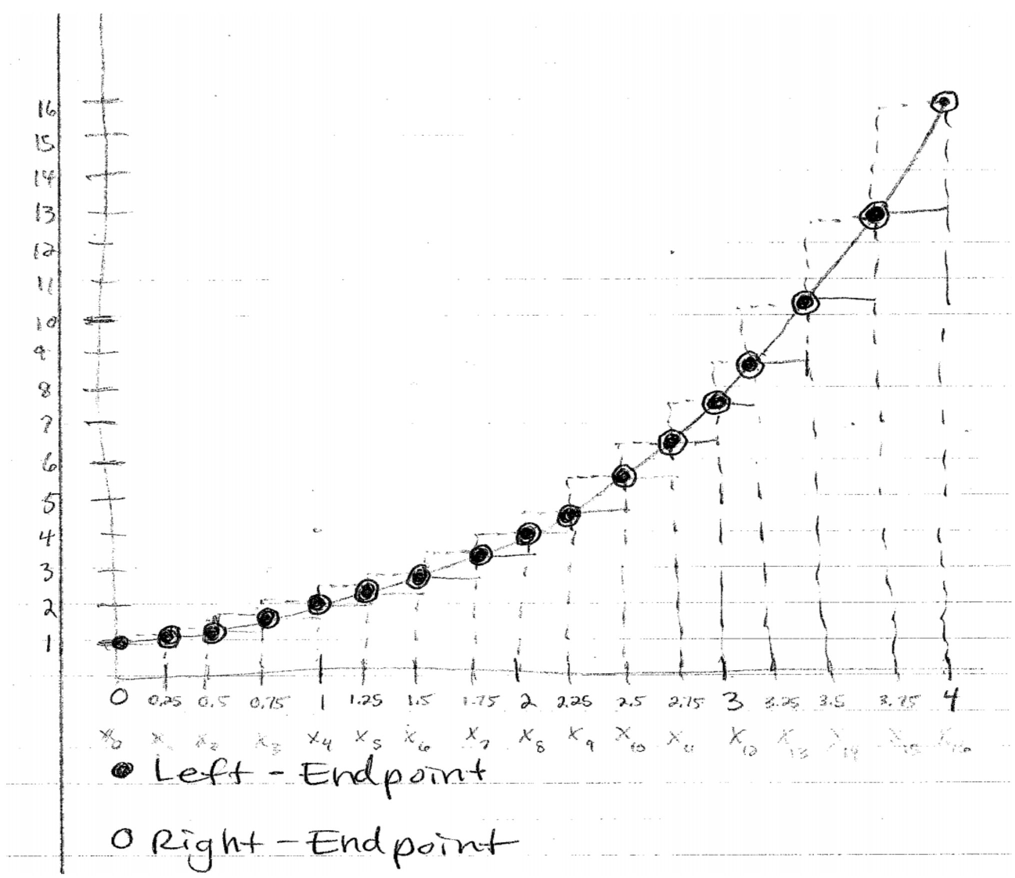
$$\text{Because } f(x_k) = 2^{x_k}\text{, then } f(x_k)^2 = 2^{x_k} \cdot 2^{x_k} = 2^{2x_k}$$

Once we have found the volumes using the left-endpoint and right-endpoint methods, determine what the volume would be if a trapezoid method were used. How are the Riemann sums for approximating area under the curve similar?

Determine the actual volume of a solid:

7. Write the integral expression that represents the volume of the solid in terms of x.

8. Find the volume of the solid as accurately as possible by evaluating the integral.



6. Left-Endpoint

k	x_k	$2^{x_k} = f(x) = y$	$f(x_k)f(x_k) = 2^{2x_k}$
0	0	1	1
1	0.25	1.19	1.41
2	0.5	1.41	2.0
3	0.75	1.68	2.83
4	1.0	2.0	4.0
5	1.25	2.38	5.66
6	1.5	2.83	8.0
7	1.75	3.36	11.31
8	2.0	4.0	16.0
9	2.25	4.76	22.63
10	2.5	5.66	32.0
11	2.75	6.73	45.25
12	3.0	8.0	64.0
13	3.25	9.51	90.51
14	3.5	11.31	128.0
15	3.75	13.45	181.02

Right-Endpoint

k	x_k	$f(x) = y = 2^x$	2^{2x}
1	0.25	1.19	1.41
2	0.5	1.41	2.0
3	0.75	1.68	2.83
4	1.0	2.0	4.0
5	1.25	2.38	5.66
6	1.5	2.83	8.0
7	1.75	3.36	11.31
8	2.0	4.0	16.0
9	2.25	4.76	22.63
10	2.5	5.66	32.0
11	2.75	6.73	45.25
12	3.0	8.0	64.0
13	3.25	9.51	90.51
14	3.5	11.31	128.0
15	3.75	13.45	181.02
16	4.0	16.0	256

Volume $= B \cdot h = s^2 \cdot h = f(x_k)f(x_k)\Delta x = f(x_k)^2 \Delta x = 2^{2x_k}\Delta x$

- Left Endpoint Volume

$0.25\,[1 + 1.41 + 2.0 + 2.83 + 4.0 + 5.66 + 8.0 + 11.31 + 16.0 + 22.63 + 32.0 + 45.25 + 90.51 + 128.0 + 181.02]$

$= 551.62 \cdot 0.25 = 137.905 \text{ in}^3$

- Right Endpoint Volume

$0.25\,[1.41 + 2.0 + 2.83 + 4.0 + 5.66 + 8.0 + 11.31 + 16.0 + 22.63 + 32.0 + 45.25 + 64.0 + 90.51 + 128.0 + 181.02 + 256]$

$= 806.62 \cdot 0.25 = 201.66 \text{ in}^3$

$137.905 < \text{Volume} < 201.66$

7. The integral expression for the volume is:

$$\int_0^4 2^{2x}\,dx$$

we learned earlier in the text that $b^x = (e^{\ln b})^x$

Therefore $2^{2x} = (e^{\ln 2})^{2x} \Rightarrow 2^{2x} = e^{2\ln 2x}$

Therefore $\int 2^{2x}\,dx = \int e^{2\ln 2x}\,dx$

8, we know that

$$\frac{d}{dx}(2^{2x}) = \frac{d}{dx} e^{2\ln 2x}\,dx$$

$$= e^{2\ln 2x} \cdot 2\ln 2 \text{ by the chain rule}$$

That means $\int 2^{2x}\,dx = \int e^{2\ln 2x}\,dx = \int e^{2\ln 2x} \frac{2\ln 2}{2\ln 2}\,dx$

$$= \frac{1}{2\ln 2}\int e^{2\ln 2x}(2\ln 2\,dx)$$

So 2^x has become 2^{2x} for volume

$$= \frac{1}{2\ln 2} e^{2\ln 2x} + C$$

Substitute 2^{2x} for $e^{2\ln 2x}$

$$= \frac{1}{2\ln 2} 2^{2x} + C$$

$$= \frac{2^{2x}}{2\ln 2}$$

now calculate

$$\left.\frac{2^{2x}}{2\ln 2}\right|_0^4$$

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$$= \frac{2^8}{2\ln 2} - \frac{1}{2\ln 2}$$

$$= \frac{256-1}{2\ln 2}$$

$$= \frac{255}{2\ln 2}$$

$$\approx 183.944$$

$$137.905 < 183.944 < 201.66$$

Left-Endpoint Volume (Under Estimate)

Right-Endpoint Sum (Over Estimate)

The Trapezoid Method is the average of the Left-Endpoint and Right-Endpoint Method

$$\frac{137.905 + 201.66}{2} = 169.783$$

This is even closer to the volume found by the integral: 183.944 in^3

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