Pre-Calculus and Calculus Module 6 Rates of Change

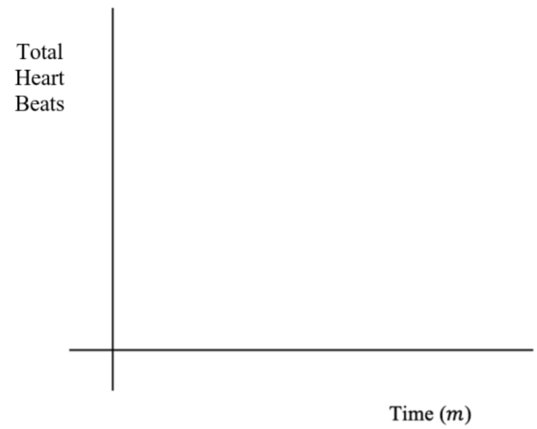
Section 6.1 Constant Rates of Change

Practice Problems 6.1

For Problem 1 and 2, use the experiment from the Lesson Notes and the information given to solve the problem.

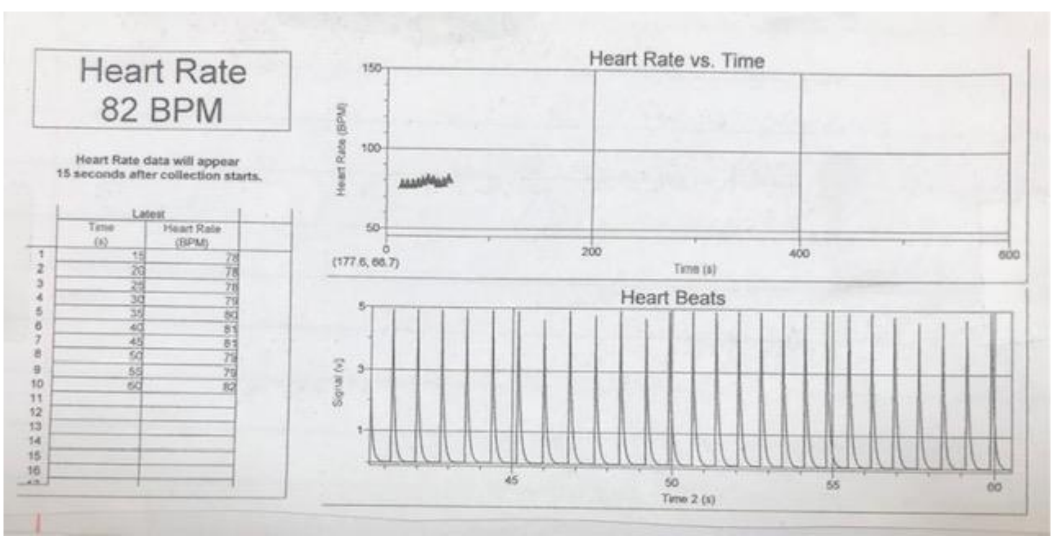
1. Record here the table of your total heart beats over the 10-minute interval from the experiment you conducted in the Looking Ahead section before Example 1 in the Lesson Notes. Draw the graph of your data.

Time (m)	Total Heart Beats
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	



2. What is the equation that would represent the total heart beats of your heart rate over time?

For Problem 3-8, use the data from Alex's heart rate to solve the problem.



Time (m)	BPM
15	78
20	78
25	78
30	79
35	80
40	81
45	81
50	79
55	79
60	82

3. Is Alex's heart rate increasing, decreasing, or constant?

4. What do you think Alex is doing?

5. Find the rate of change from 15 seconds to 60 seconds?

6. If the rate of change were 0, what would that mean for Alex's activity?

7. What would the graph look like if the rate of change were 0?

8. Is it possible for the heart rate to be constant over a long period of time? Does it seem likely or highly unlikely? Why or why not?

For Problem 9-20, use the given information to solve the problem.

Suppose Alex walks through a doorway after taking his heart rate; he gets up to get a drink and pushes the door open at $t = 0$ seconds and it slams at $t = 9$ seconds later. It is heavy and has a hinge that causes it to close slowly. The number of degrees (d), from the closed position is $d = 188t \cdot 3^{-t}$ for $0 \leq t \leq 9$.

9. Use a calculator to draw the graph of the function. Explain why the degrees are increasing and then decreasing.

10. At what point in time does the door appear to start closing?

11. At what degree is the door from the closed position at 1 second?

12. At what degree is the door from the closed position at 2 seconds?

13. Find the average rate of change between 1 and 2 seconds?

Let us investigate points close to 5 seconds to find the instantaneous rate of change at the instant of 5 seconds.

14. How many degrees is the door from the closed position at $t = 5$ seconds?

15. How many degrees is the door from the closed position at $t = 5.1$ seconds?

16. How many degrees is the door from the closed position at $t = 5.01$ seconds?

17. What is the average rate of change in degrees per second from 5 to 5.1 seconds?

18. What is the average rate of change in degrees per second from 5 to 5.01 seconds?

19. As the seconds get closer and closer to 5, what happens to the average rate of change?

20. What does the limiting value of the degrees per second seem to be at 5 seconds? What does the negative sign tell us about the direction of the door? What is the approximate instantaneous rate of change at 5 seconds?

Section 6.2 Changing Rates of Change

Practice Problems 6.2

For Problem 1-5, use the information given to solve the problem.

1. If exponential functions increase at an increasing rate, does that mean their inverse (logarithmic functions) decrease at a decreasing rate?

2. If a function is linear and has values $g(5) = 10$ and $g(15) = 18$, what is $g(20)$? Use the add-add property to solve the problem.

3. If a function is an exponential function and has the values listed in the table below, find $g(15)$ and $g(20)$. Use the add-multiply property to solve the problem.

$\boldsymbol{x}$	$\boldsymbol{g(x)}$
5	16
10	512
15	
20	

4. Given the table of values below, is the pattern the add-add property, add-multiply property, or multiply-multiply property?

$\boldsymbol{x}$	$\boldsymbol{g(x)}$
5	1,250
10	20,000
15	101,250
20	320,000

5. Is the function for the table in Problem 4 linear, exponential, or a power function?

For Problem 6-8, use the information from the previous problem given to solve the problem.

6. What is the linear equation for Problem 2?

7. What is the exponential equation for Problem 3?

8. What is the power equation for Problem 4?

For Problem 9-14, use the information given to solve the problem.

9. Name three functions that are power functions.

10. Using the quadratic equation $p(x) = 3x^2 - 2x + 4$, find the differences between the consecutive y-values. These are called first differences. What are the first differences increasing by?

$\boldsymbol{x}$	$\boldsymbol{p(x)}$
1	5
2	12
3	25
4	44
5	69
6	100

11. Given the table of values from Problem 10, what can be said about the first differences?

12. Find the output differences between the first differences. These are called the second differences. What type of equation has equivalent second differences?

13. What type of equation has constant first differences?

14. What type of equation has constant third differences?

For Problem 15-20, use the information below to find the given limit using substitution.

The speed of a moving body at a given instant of time is the instantaneous speed (when the elapsed time seems to be zero at that point). Limits help us calculate instantaneous speed. Let us practice with limits.

15. $\lim_{x \to 2} 2x^3(3x+1)$

16. $\lim_{x \to \frac{1}{3}} (3x-7)$

17. $\lim_{x \to -3} (6x-9)^{\frac{2}{3}}$

18. $\lim_{x \to 4} \sqrt{x^2+9}$

19. $\lim_{y \to 1} \frac{y^2+2y-1}{y+8}$

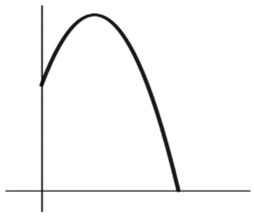
20. $\lim_{x \to -2} (x^3+x^2-x-10)$

Section 6.3 Distance-Time Graphs

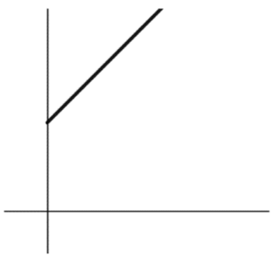
Practice Problems 6.3

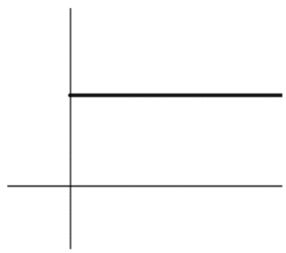
For Problem 1 and 2, use the information given to solve the problem.

1. Describe a situation involving a motion detector that has the following graph.



2. Which graph shows the car stopped and which graph shows car traveling at a constant speed? Explain why.





For Problem 3-7, use the functions that model an object's motion to answer the following questions:

a) $d(t) = 5.9t + 3$

b) $d(t) = -4.7t + 8$

c) $d(t) = 5.9t + 2.6$

d) $d(t) = -4.7t$

e) $d(t) = 4.1t$

3. Which object has the slowest speed?

4. Which object starts farthest from the motion detector?

5. Which objects are traveling at the same speed?

6. Which objects are traveling at the same velocity?

7. Which object(s) start(s) closest to the motion detector?

For Problem 8-10, use the given data to solve the problem.

x	y
-3	-49
-2	-22
-1	-5
0	4
1	2
2	-13
3	-34

8. Over what measured intervals does this graph appear to…
a) … increase? b) … decrease?

c) … change from increasing to decreasing?

9. Find the average rate of change from $(0, 2]$.

10. Find the average rate of change from $[1, 3]$.

For Problem 11-14, use the given information to solve the problem.

The velocity for a moving object over the interval $[0, 6]$ is $v(t) = t^2 - 5t + 8$ in feet per second.

11. Complete the table.

t (s)	$v(t)$ $(\frac{ft}{s})$
0	
1	
2	
3	
4	
5	
6	

12. Over what interval is the velocity increasing?

13. Over what interval is the velocity decreasing?

14. What is the velocity of the moving object at $t = 2.5$?

For Problem 15-17, use the table that shows position of an object in meters over seconds to solve the problem.

$\boldsymbol{t}$	$\boldsymbol{x(t)}$
0	0.0
3	0.016
6	0.039
9	0.066
12	0.101
15	0.144
18	0.185
21	0.213

15. Over what interval is the maximum displacement?

16. Over what interval is the minimum displacement?

17. What is the total displacement from $[0, 21]$?

For Problem 18-20, given the situation, solve the problem.

18. Ezekiel travels 75 mi./h. for 4 hours. How far did he go?

19. Jonas travels 270 miles at 60 mi./h. How long did he travel?

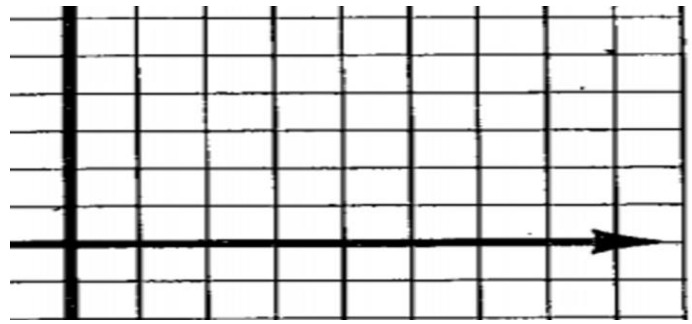
20. Braden traveled 130 miles in 2 hours. How fast did he travel?

Section 6.4 Slope and the Secant Line

Practice Problems 6.4

For Problem 1-4, use the graph of $f(x) = \sqrt{2x+1}$ to solve the problem.

1. Draw the graph of $f(x) = \sqrt{2x+1}$ and answer the following questions.



a) Over what interval is it increasing?

b) Over what interval is it decreasing?

2. Use colored pencils to draw the secant lines over the following intervals:

a) Red $[1, 4]$ b) Blue $[1, 3]$ c) Yellow $[1, 2]$

3. Find the average rate of change over the following intervals:

a) $[1, 4]$ (Slope of the red line)

b) $[1, 3]$ (Slope of the blue line)

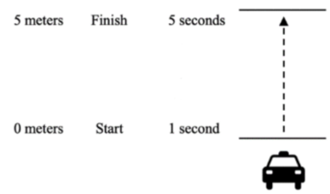
c) $[1, 2]$ (Slope of the yellow line)

Use the exact values to calculate the slope. Convert the answer to an approximation to the thousandths place.

4. Name another way to find the average rate of change from $[1, 4]$ that is different from a) in Problem 3.

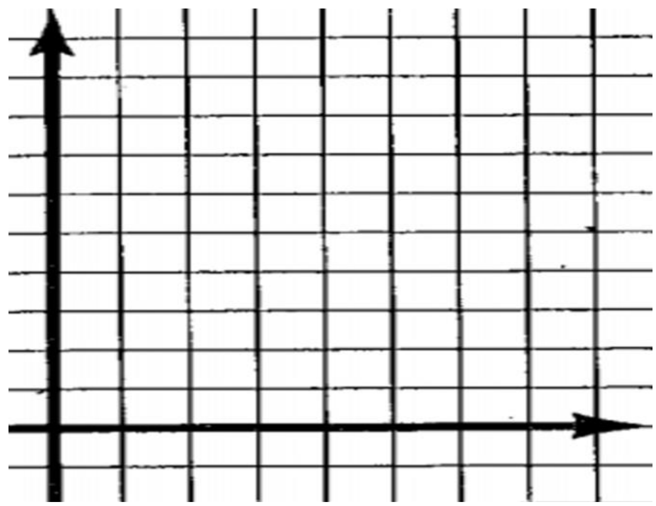
For Problem 5-10, use the given information to solve the problem.

A remote-control car was started at the start line and increased in speed over the finish line 5 meters from the start. A team of 5 people with stopwatches stood along the side and marked where the car passed at each second.



5. Graph the table of results as a continuous function. Let $\boldsymbol{x}$ represent the time in seconds and let $\boldsymbol{y}$ represent the meters from the start.

Time x (s.)	**Distance y (m.)**
0	0
1	0.2
2	1
3	2.3
4	3.6
5	6.1



6. How can you tell from the graph that the speed is increasing?

7. Find the average velocity of the remote-control car over the interval $[0, 5]$.

8. Find the average velocity over the following intervals:

a) $[1, 2]$

b) $[2, 3]$

c) $[3, 4]$

d) $[4, 5]$

e) $[0, 1]$

9. What is the average of the velocities in Problem 8?

10. How do the average velocity of the entire interval in Problem 7 and the average of velocities of the five intervals in Problem 9 compare?

For Problem 11-13, use the given information and table to solve the problem.

A cyclist starts at rest and travels at one second intervals. Let $\boldsymbol{t}$ represent the time in seconds and $\boldsymbol{d}$ represent the distance from the start.

$\boldsymbol{t}$ (**s.**)	0	1	2	3	4	5	6
$\boldsymbol{d}$ (**m.**)	0	3.2	7.3	12.8	19.2	25.6	32.6

11. What is the average speed of the cyclist from 0 to 2 seconds.

12. Show two ways to estimate the speed of the cyclist from 0 to 3 seconds.

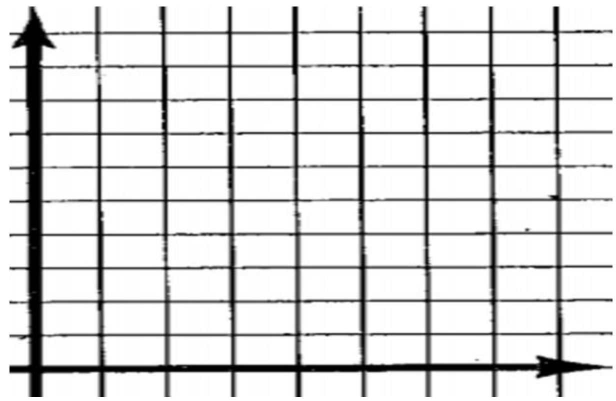
13. What is the average velocity of the cyclist from 0 to 3 seconds?

For Problem 14-20, use the information given to solve the problem.

14. Sketch the graph of $f(x) = \sec x$ from $[0,\ 1.5]$.

15. Complete the table for $y = \sec x$. Use the trace button on the calculator and set the steps accordingly.

x	$\sec x$
0	
0.3	
0.6	
0.9	
1.2	
1.5	



16. What is the slope for $f(x) = \sec x$ over $[0, 1.5]$?

17. What is the slope of $f(x) = \sec x$ over the following intervals?

a) $[0, 0.3]$

b) $[0.3, 0.6]$

c) $[0.6, 0.9]$

d) $[0.9, 1.2]$

e) $[1.2, 1.5]$

18. What is the average of the slopes of the intervals found in Problem 17?

19. What is another means of expressing $\sec x$ using another trigonometric function?

20. What is the slope of the trigonometric function in Problem 19 over the interval $[0,\ 1.5]$?

Section 6.5 Average Rates of Change and Velocity

Practice Problems 6.5

For Problem 1-4, determine if the average rate of change over the interval is positive or negative for the function.

1. $f(x) = x^3$ over the interval $[-3, -2]$

2. $f(x) = x^3$ over the interval $[2, 3]$

3. $g(x) = e^x$ over the interval $[-1, 1]$

4. $h(x) = 3^x$ over the interval $[0, 4]$

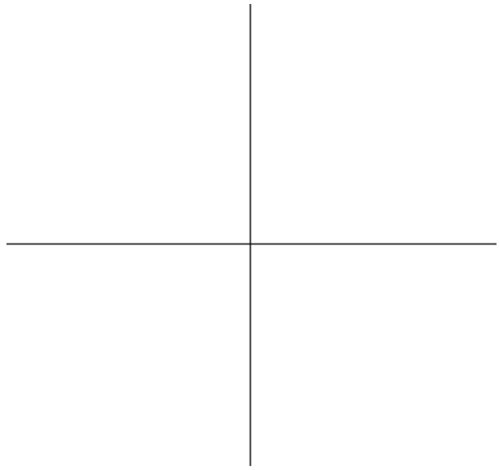
For Problem 5-7, let the distance function for the toy car be $d(t) = 10.17t^2$ in terms of feet traveled once the car has started over time t measured in seconds. Find the average velocity over the given intervals.

5. $[4, 4.1]$

6. $[4, 4.01]$

7. $[4, 4.001]$

For Problem 8-10, draw the graph of $f(x) = e^x$ and answer the questions that follow. Let the growth be represented by feet per second.



8. What is the average rate of change over the interval $[0, 0.1]$?

9. What is the average rate of change over the interval $[0, 5]$?

10. When does the slope seem to be the smallest? When does the slope appear to be the largest?

For Problem 11-20, use the given information to solve the problem.

Garrett is rock climbing and his foot drops as a rock breaks off and falls to the lake below. Dense objects that are solid fall to earth so that the displacement downward is $f(t) = 16t^2$ feet in t seconds. An object is considered to be in free fall if the only force acting on it is gravity.

11. What is the average speed the rock falls in the first 2 seconds, $[0, 2]$?

12. What is the average speed the rock falls in the first 4 seconds, $[0, 4]$?

13. What is the average speed of the rock between 2 and 4 seconds?

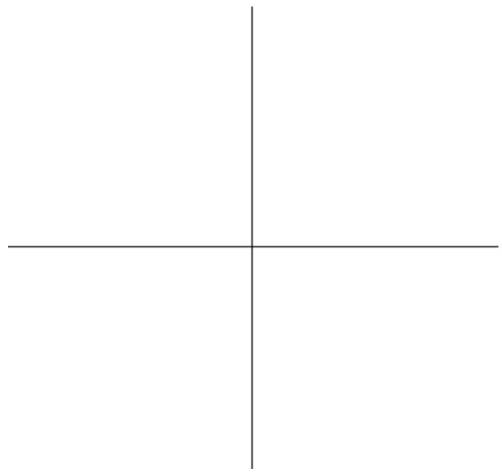
14. Why can't we use the average rate of speed $\frac{\Delta f(x)}{\Delta x}$ to find the exact speed of the rock between $t = 2\ s$ and $t = 2\ s$? In other words, the exact speed of the rock when $t = 2\ s$.

15. Use "Lists and Spreadsheets" on the graphing calculator to complete a table of values for the rock's speed close to 2 seconds.

$\boldsymbol{t}$	$\boldsymbol{f(t)}$
2	
2.01	
2.001	
2.0001	
2.00001	
2.000001	

16. Based on the table, find the speed of the rock at $t = 2s$.

17. Sketch the graph of $f(t) = 16t^2$. Use this graph to solve Problem 18-20.



18. Find $\lim_{t \to 2^-} 16t^2$

19. Find $\lim_{t \to 2^+} 16t^2$

20. If instead the displacement downward, f, is taken to be the height of the rock above the ground, then f must be written as $f(t) = h_o - 16t^2$ or $f(t) = -16t^2 + h_o$. What would the value h_o represent? Why must $16t^2$ be subtracted from h_o rather than added to it?

Section 6.6 Average Rates of Change

Practice Problems 6.6

For Problem 1-10, let us look back at the toy push car problem to solve the problem.

We used values of 4.1, 4.01, and 4.001 as t approached 4.0 seconds from the right. Now we will investigate as t approaches 4.0 seconds from the left using the distance function $d(t) = 10.17t^2$.

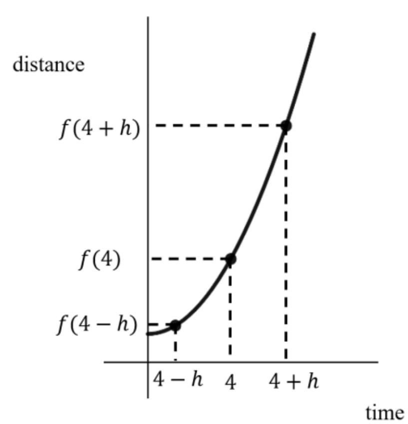
1. Find the average velocity of the car over the interval $[3.9, 4]$.

2. Find the average velocity of the car over the interval $[3.99, 4]$.

3. Find the average velocity of the car over the interval $[3.999, 4]$.

4. When we found the average velocity as time got closer to 4 seconds from the right, the average velocity got smaller. As you approach 4 seconds from the left, the average velocity gets larger. Why does this make sense when $h > 0$?

5. The graph below depicts intervals of time (h) before and after 4 seconds. What is the change in time over the interval $[4 - h, 4]$?



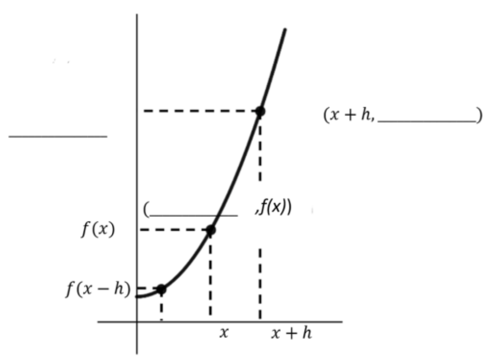
6. What is the change in distance over the interval $[4-h, 4]$?

7. What is the average velocity over the interval $[4-h, 4]$?

8. Use the expression found in Problem 7 to find the average velocity over the interval $[4-h, 4]$ for the toy push car.

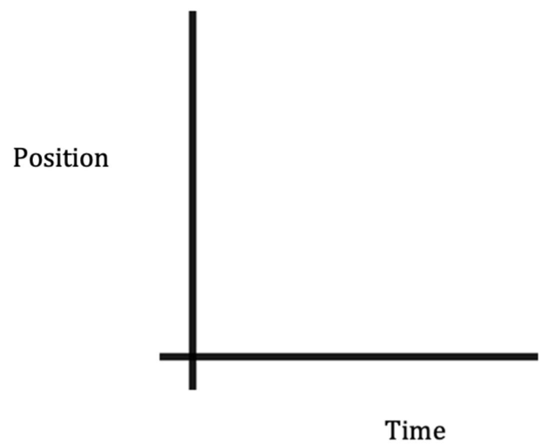
9. How does the average velocity we found in Problem 8 compare with the average velocity of the car over the interval $[4, 4+h]$ found in Example 1 of the Lesson Notes?

10. Fill in the blanks to complete the generalized graph below.



For Problem 11-20, complete the time-position and time-velocity graphs for the motion of a remote-controlled car.

For Problem 11-15, complete the time-position graph.



11. Firstly, the remote-controlled car drives away from the motion detector at a slow constant rate for 10 seconds.

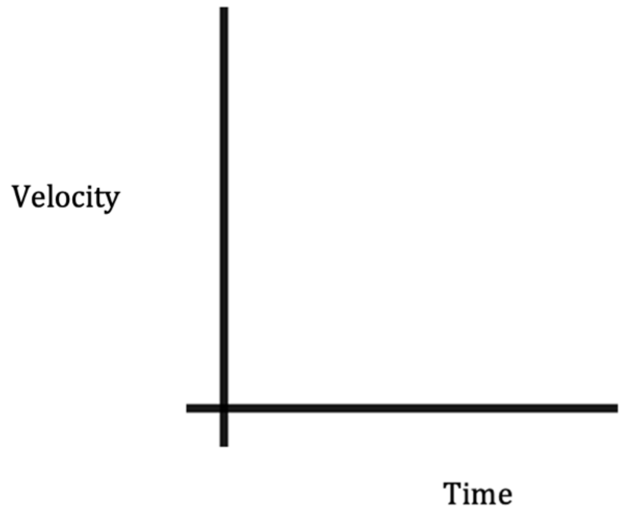
12. Then the remote-controlled car idles for 10 seconds.

13. Next the remote-controlled car drives back toward the motion detector at the same slow constant rate for 5 seconds.

14. Finally, the remote-controlled car drives away from the motion detector at a faster, but still constant, rate for 10 seconds.

15. And lastly the remote-controlled car stops for the last 5 seconds.

For Problem 16-20, complete the time-velocity graph.



16. Over what interval of time is the velocity zero?

17. Explain how there can be zero velocity.

18. When is the velocity positive?

19. When is the velocity negative?

20. Why is velocity on the time-velocity graph on the x-y plane represented by horizontal lines at each interval?

Section 6.7 Slope and the Tangent Line

Practice Problems 6.7

For Problem 1-15, use the given information to solve the problem.

Let us suppose you are building a box to store all of your math games. There is no lid on the box. It needs to have a volume of 81 cubic feet. The material for the square base of the box is \$3 per square foot. The material to cover the four sides of the box is \$2 per square foot; it is not as expensive as the base because it does not need to be as durable.

1. Draw the box and label its dimensions. Let x be the length of the sides of the base and h be the height.

2. Write an expression for the cost to cover the base of the box in terms of x.

3. Write an expression for the cost to cover the sides of the box in terms of x and h.

4. Write an expression for the total cost to cover all sides of the box in terms of x and h.

5. Use the fact that the volume of the box is 81 cubic feet to solve for h in terms of x.

6. Substitute the value from Problem 5 in for h in the expression from Problem 4 so the cost of the box is now expressed in terms of x only (it is your only variable).

7. Set the viewing window on the calculator so that the y-scale is the interval $[-10, 800]$. Use the trace button to find the minimum cost of the box and the minimum length at that point.

8. Find the height of the box given the minimum length found in Problem 8.

9. What does the slope of the tangent line at the minimum point appear to be?

10. Will any maximum or minimum point on a curve have the same slope for the line tangent to it that point? Why or why not?

11. Name two other lengths for bases and their total cost that would result in a box with 81 cubic feet of space inside.

12. What is the cost of the box when the length of the base is 6 feet?

13. Calculate the height of the box when the length of the base is 6 feet?

14. Calculate the volume of the box when the base is 6 feet? What do you expect the volume to be?

15. Substitute x and h from Problem 12-14 in the formula for the cost of the box to check your work.

For Problem 16-20, select the correct answer.

16. If $\lim_{x \to c} f(x) = f(c)$, then f is ______________________ at $x = c$.

a) continuous
b) increasing
c) decreasing
d) zero

17. The tangent to a graph at point $(c, f(c))$ is the limit of the ____________________ line drawn from $(c, f(c))$ to $(x, f(x))$.

a) sine
b) cosine
c) secant
d) cosecant

18. For the definition of limit, L is equal to $\lim_{x \to c} f(x)$ if for any value $\epsilon > 0$, there is a value $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - \mathrm{L}| <$ ________.

a) c
b) x
c) δ
d) ϵ

19. If the expression E has the indeterminant form of $\frac{0}{0}$ at c, and has a limit of L as x approaches c, then the function $f(x) = \mathrm{E}$ has a _____________________ discontinuity at the point (c, L).

a) undefined
b) indeterminant
c) non-removable
d) removable

20. If $\lim_{x \to \infty} f(x) = \mathrm{L}$, then there is a _____________________ asymptote at the line $y = \mathrm{L}$.

a) vertical
b) horizontal
c) zero
d) indeterminant

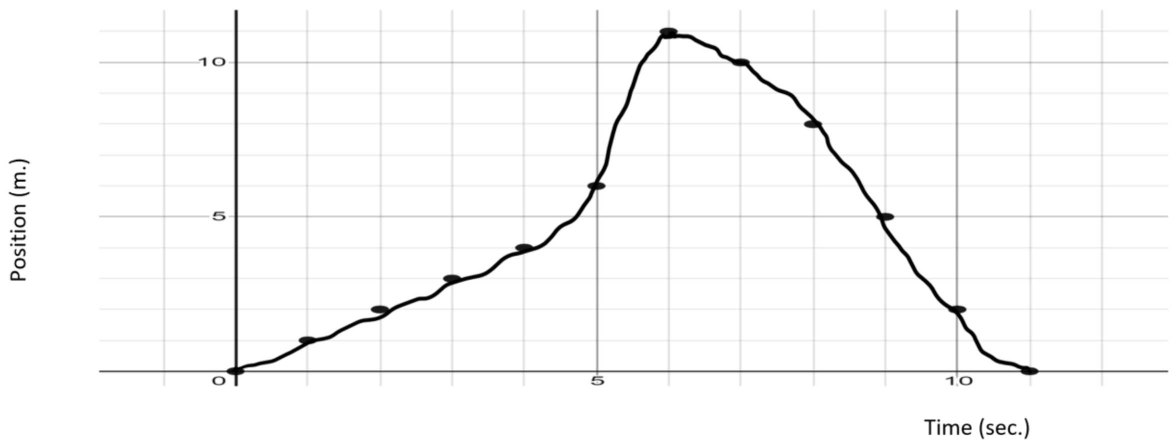
Section 6.8 Instantaneous Rates of Change and Velocity

Practice Problems 6.8

Given are definitions for instantaneous velocity and average velocity.

Instantaneous Velocity: The velocity of an object at one moment in time.
Average Velocity: The velocity of an object over an extended period of time.

For Problem 1-14, use the position-time graph for the motion of a remote-control car to solve the problem.



1. Over what interval is the slope positive? What does this mean about the position of the car?

2. The interval $(6,11]$ has a negative slope. What does this mean about the velocity of the car?

3. What is the instantaneous velocity of the car when it changes directions at approximately 6 seconds?

4. The car moves the fastest when the slope is the steepest, which is between 5 and 6 seconds. What is the average velocity during this time interval?

5. Find the average velocity over the interval $[1, 3]$ and $[3, 5]$.

6. Which interval in Problem 5 best approximates the instantaneous velocity at 2 seconds?

7. Find the average velocity over the interval $[0, 2]$ and $[2, 4]$. Which is the best approximation for the instantaneous velocity of the car at 2 seconds?

8. At what interval is the velocity constant? How do you know?

9. At what interval is the velocity increasing the most? How do you know?

10. Compare the velocity over the intervals $[5, 6]$ and $[7, 10]$.

11. Sketch the velocity-time graph over $[0, 4)$.

12. Sketch the velocity-time graph over $(4, 6)$.

13. Sketch the velocity-time graph over $(6, 8)$.

14. Sketch the velocity-time graph over $(8, 10)$.

For Problem 15-20, sketch the graph for a function that demonstrates the given situation.

15. is continuous at $x = 3$

16. has a left-sided limit of -1 and a right-sided limit of 3 as x approaches 0

17. has a vertical asymptote at $x = -2$

18. has a step discontinuity at $x = 5$

19. has a removable discontinuity at $x = 4$

20. is discontinuous at $x = 2$ but has a limit at x approaches 2 and a value at $f(2)$

Section 6.9 Instantaneous Rates of Change for Functions

Practice Problems 6.9

For Problem 1-3, let $g(x)$ be equal to $\cos x$ to solve the problem.

1. Find the average rate of change from $x = 0$ to $x = \frac{\pi}{2}$. Keep your answer an exact value.

2. Find the average rate of change from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$. Keep your answer an exact value.

3. Find the average rate of change from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$. Keep your answer an exact value.

For Problem 4-6, let $h(x)$ be equal to $x^3 + 4$ to solve the problem. The function name of h is different from the change of h over the interval.

4. Find the average rate of change from $x = 3$ to $x = 5$. Keep your answer an exact value.

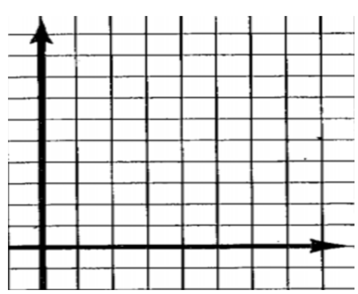
5. Write the expression for the average rate of change over the interval 3 to $3 + h$.

6. Find the derivative of $h(x)$ using the expression for Problem 5.

For Problem 7-13, use the given information to solve the problem.

A space probe is sent to Mars and its travel is recorded in kilometers. The distance, $d(x)$, from the planet, is modeled by the equation $d(x) = x^2 - 6x + 22$ in which x is minutes.

7. Sketch a graph of $d(x)$.

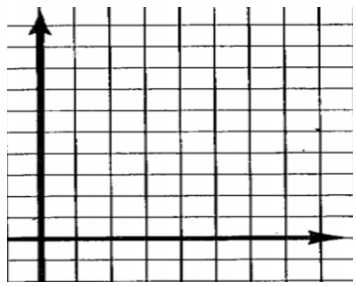


8. What is the average rate of change from 5 to 5.1 seconds?

9. The average rate of change for the function d from 5 to x is shown as follows:

$$r(x) = \frac{d(x) - d(5)}{x - 5}$$

Simplify the function and sketch its graph.



10. Where is the removable discontinuity for $r(x)$?

11. Find the derivative of d at $x = 5$ by taking the limit of $r(x)$ as x approaches 5.

12. Find the equation of the line that goes through the point $(5, d(5))$ and has the slope of the velocity in Problem 11. Use the point-slope form equation of a line.

13. Graph the line found in Problem 12 on the same graph as function d. What do you notice about the line?

For Problem 14-20, use the given information to solve the problem.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Use the equation above to find the instantaneous rate of change for $f(x)$ with respect to x at $x = c$.

14. Find $f'(4)$ for $f(x) = x^2 + x - 6$.

15. Where is there a removable discontinuity for the function $r(x) = \frac{f(x)-f(4)}{x-4}$?

16. Find the equation of the line that goes through the point $(4, f(4))$ and has the derivative (slope) at $x = 4$, which is limit of $r(x)$ as x approaches 4.

17. Graph the function f and the tangent line $y = 9x - 22$ on the same graph. At what point do they intersect?

18. Find $f'(c)$ when $f(x) = x^2 - 8x - 7$ and $c = -3$.

19. Find $f'(c)$ when $f(x) = -9$ and $c = -2$.

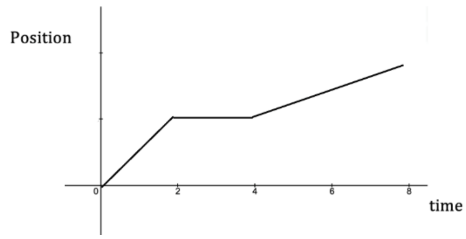
20. Find $f'(c)$ when $f(x) = x^3 + 2x^2 - x - 2$ and $c = 1$.

Section 6.10 Velocity and Position Graphs

Practice Problems 6.10

For Problem 1 and 2, use the graph in Problem 1 to solve the problem.

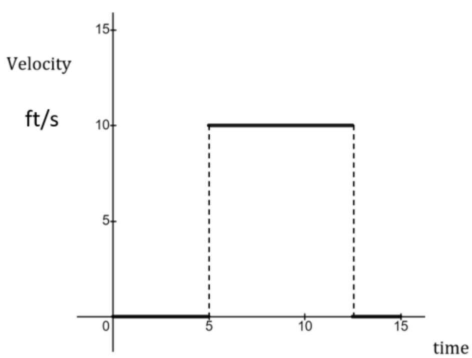
1. Sketch a velocity graph to match the position-time graph.



2. Write a possible scenario to describe the graph in Problem 1.

For Problem 3-6, use the graph in Problem 3 to solve the problem.

3. Sketch a position-time graph to match the velocity graph.

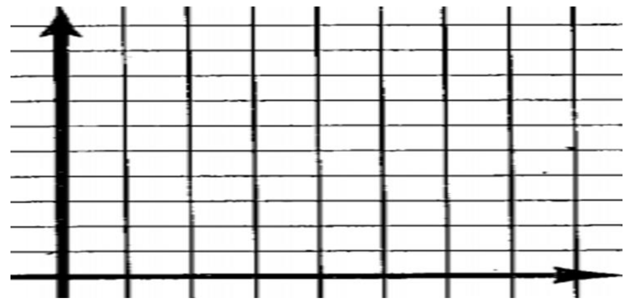


4. Write a possible scenario to describe the graph in Problem 3.

5. Graph the velocity of an object in meters per second with respect to time using the given piecewise function. Then answer the questions that follow.

$$v(t) = \begin{cases} 15 & 0 \leq t \leq 20 \\ 0 & 20 < t < 40 \\ 5 & 40 \leq t \leq 60 \end{cases}$$

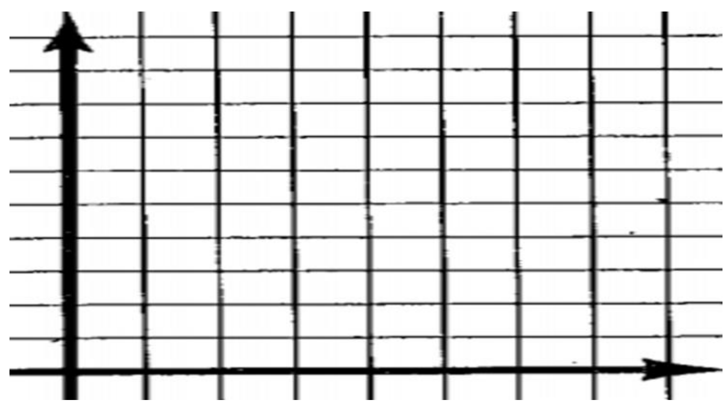
a) Draw a velocity graph showing the velocity from 0 to 60 seconds. The velocity is in meters per second.



b) What is the total area under the velocity curve?

c) What does the total area under the velocity curve represent?

6. Draw a distance-time graph that conforms to the velocity from 0 to 60 seconds in Problem 5.



a) When $t = 20$, what is $d(t)$?

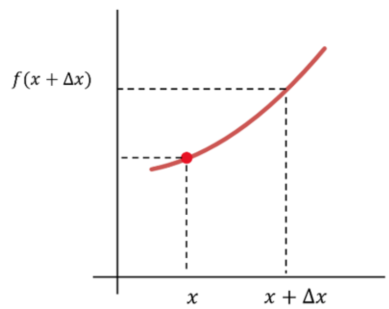
b) When $t = 60$, what is $d(t)$?

c) What is the slope over the interval $[0, 20]$?

d) What is the slope over the interval $[40, 60]$?

e) Where do you see the slopes on the velocity graph?

For Problem 7-15, use the given graph to solve the problem.



7. Label $f(x)$ on the graph.

8. Circle the point on each graph where the derivative is found.

9. Highlight the change of $f(x + \Delta x) - f(x)$ on the y-axis of the graph in blue.

10. What letter and symbol represent h (change in x) on the graph. Highlight this in yellow.

11. Recall from Algebra that $m = \frac{\Delta y}{\Delta x}$. Use this to write the slope of the graph.

12. What is the limit of the slope of the graph called.

13. Write the limit of the slope of the graph.

14. Write the equation for the derivative in terms of the limit of the slope of the graph.

15. Substitute c for x and h for Δx to get another standard form variation of the derivative function.

For Problem 16-20, use the information in Problem 16 to solve the problem.

16. Let $g(x) = x^3$ and $c = 4$. Let $h = 0.01$. Find $\frac{g(c+h)-g(c)}{c+h-c}$.

17. Using the information from Problem 16, find $\frac{g(c-h)-g(c)}{c-h-c}$.

18. Using the information from Problem 16, find $\frac{g(c+h)-g(c-h)}{(c+h)-(c-h)}$.

19. What do you think is the exact value of the derivative of $g(x) = x^3$ when $c = 4$?

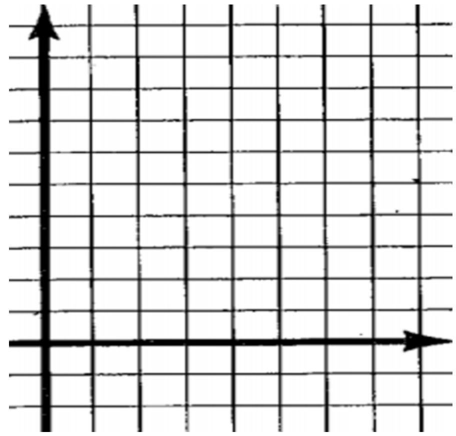
20. Find the $f'(x) = \lim_{h \to 0} \frac{g(x+h)-g(x-h)}{(x+h)-(x-h)}$ to confirm the derivative in Problem 19.

Section 6.11 Distance and Velocity are Related

Practice Problems 6.11

For Problem 1-6, let the velocity function be $v(t) = 2t$.

1. Graph the velocity function over the interval $[0, 4]$.



2. Find $v(1)$, $v(2)$, and $v(t+h)$.

3. Let $t + h$ be any value for x. Find $v(x)$.

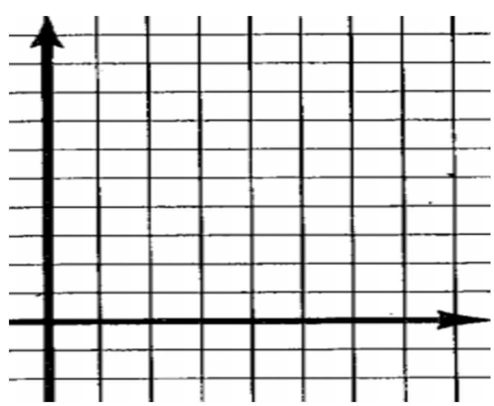
4. Find the area under the curve over the interval $[0, 4]$. Use the equation $A = \frac{1}{2} b \cdot h$ because the area is a triangle. What is the base in terms of t? What is the height in terms of $v(t)$?

5. Let x be any point on the x-axis over the interval $[0, 4]$. Write an expression for area under the curve.

6. What does the area under the velocity curve represent?

For Problem 7-12, let the distance function be $d(t) = t^2$.

7. Graph the distance function over the interval $[0, 4]$.



8. Find $d(1)$, $d(2)$, and $d(t+h)$.

9. Let $t+h$ be any value for x. Find $d(x)$.

10. What is the total distance at 4 seconds?

11. Find $d'(t)$.

12. How does the velocity relate to the distance?

For Problem 13-20, let $d(t) = 5t^2$ and $v(t)$ be in meters per second to solve the problem.

13. Find the average velocity from $t = 3$ to $t = 3.1$.

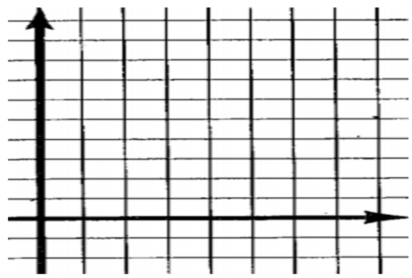
14. Find the average velocity from $t = 3$ to $t = 3.01$.

15. What is the change in time from $t = 3$ to $t = 3 + h$?

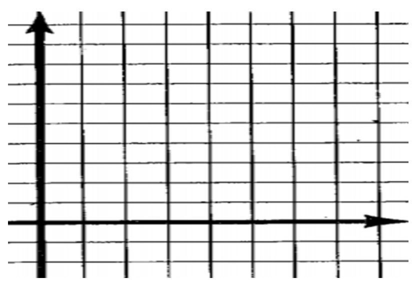
16. What is the change in distance from $t = 3$ to $t = 3 + h$?

17. What is the average velocity (the ratio of the change in distance over the change in time) from $t = 3$ to $t = 3 + h$?

18. Draw the distance-time graph for function d from $t = 3$ to $t = 3 + h$.



19. Sketch the graph of $f(x) = 12 - 2x$ from $[0, 6]$. Label a point x on the x-axis in the interval. Shade the area under the curve to the left of point x. Write the expression for the area under the curve over the interval $[0, x]$. (Hint: The area shaded is a trapezoid.)



20. Find an expression for the area under the curve for $f(x) = 12 - 2x$ that is non-shaded over the interval $[x, 6]$. (Hint: The non-shaded area is a triangle.)

Section 6.12 Graphs of Functions and their Derivatives

Practice Problems 6.12

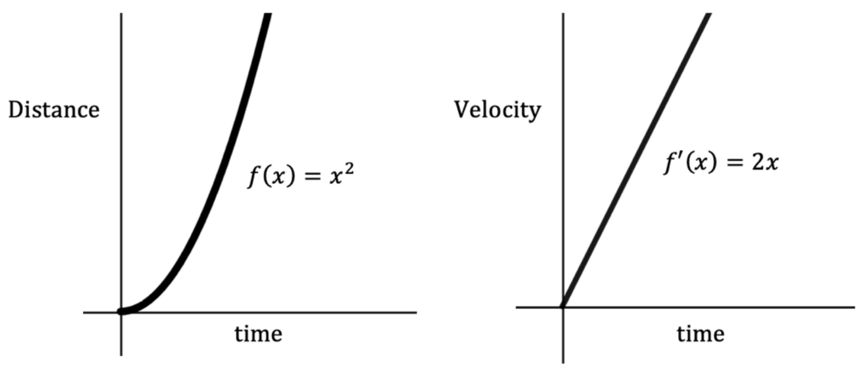
For Problem 1-3, let $f(x) = x^2$ be the distance function and $f'(x) = 2x$ be the velocity function and fill in the blanks.

1. The distance function is the area under the ____________________ curve and the velocity function is the ____________________ of the distance function.

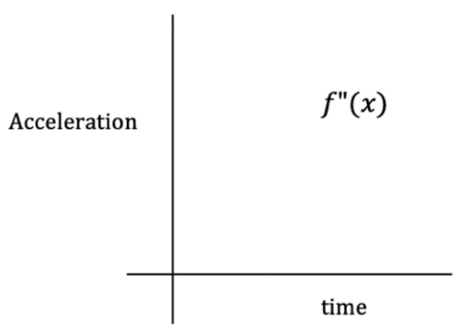
2. To change a ____________________ function to a distance function, find the general expression for the ____________________ under the curve and graph that.

3. To change a distance function to a ______________ function, take the ____________________ of the distance function.

For Problem 4-8, use the given information to solve the problem.



4. If acceleration is the change in velocity over the elapsed time, draw the acceleration graph that goes with the graphs above. This is called f double prime of x and is the second derivative.



5. Let $f'(x) = 2x$. Find $f"(x)$ algebraically using $a = \frac{\Delta v}{\Delta t}$ and $\lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{x+h-x}$ where a is acceleration.

6. Let $f(x) = x^2$. Find the derivative of $f(x)$ at $x = 5$ using $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$ and $x = a$. (Hint: a should be 5 at some point.)

7. Let $f'(x) = 2x$. Find $f'(x)$ when $x = 5$. Did you get the same answer as in Problem 6?

8. Find the derivative $f(x) = x^2$ at $x = c$ (some constant) using $f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$.

For Problem 9-20, use the given information to solve the problem.

Solve the limits below to find the amount of each ingredient for your Calculus Cup Cake. Then follow the directions to cook your Calculus Cake in a Cup.

9. $\lim\limits_{x \to 0}(3x + 2)$ Tablespoons of Flour

10. $\lim\limits_{x \to 4} \frac{x-3}{x^2-12}$ Teaspoon of Baking Powder

11. $\lim\limits_{x \to 0} 1$ Teaspoon of Brown Sugar

12. $\lim\limits_{x \to -2}(3x^4 + 10x - 27)$ Tablespoon of White Sugar

13. $\lim\limits_{x \to 2^-} f(x)$ if $f(x) = \begin{cases} 4 - x & x \le 2 \\ \frac{5}{x} - 9 & x > 2 \end{cases}$ Teaspoons of Cocoa Powder

14. $\lim\limits_{x \to -4} \left| \frac{x^2+2x-3}{x-1} \right|$ Pinch of Salt (Take the absolute value of the limit)

15. $\lim\limits_{x \to 0} \frac{1}{4}$ Teaspoon of Baking Powder

16. $\lim\limits_{x \to -5} (x+4)^8 + 1$ Tablespoons of Milk

17. $\lim\limits_{x \to 3} \frac{x^2-3x+18}{2x^2-6x+24}$ Teaspoons of Oil

18. $\lim\limits_{x \to 2} \frac{x-1}{x^2}$ Teaspoons of Vanilla Extract

19. Mix all the dry ingredients. Mix all the wet ingredients. Pour the oil inside a ceramic or microwaveable cup. Pour the mixed batter into the cup. Microwave the cup for 50-65 seconds until it is done. Top the final product with ice cream, whip cream, and/or hot fudge! Enjoy!

20. Try to write your own recipe using limits! The sky is the limit for this assignment.

Section 6.13 Distance, Velocity, and Acceleration

Practice Problems 6.13

For Problem 1-8, use the given information to solve the problem.

The takeoff of a Boeing 747 airplane will allow you to investigate the relationships between distance, velocity, and acceleration. Below is the time and acceleration as the plane moved down the runway.

time (sec.)	acceleration
0	0.13
1	0.13
2	0
3	0.13
4	0.13
5	0.13
6	0
7	0.9
8	1.58
9	2.58
10	2.58
11	2.3
12	3.34
13	2.95
14	3.092
15	2.697
16	2.443
17	2.58
18	2.57
19	3.98
20	3.47

time (sec.)	acceleration
21	3.01
22	2.69
23	2.57
24	2.95
25	3.08
26	3.08
27	3.21
28	2.69
29	1.93
30	3.21
31	2.95
32	2.31
33	1.67
34	3.21
35	2.05
36	3.34
37	3.59
38	4.11
39	4.49
40	4.80

1. What is the interval between the time data?

2. What is the maximum acceleration?

3. What seems to be the general trend of the data?

Enter the data into the "Lists and Spreadsheets" page of the graphing calculator. Let Column A be time and Column B be acceleration. Create a scatterplot on the "Data and Statistics" page with the x-axis being time and the y-axis being acceleration. Call Column A "time" and call Column B "acc." Change the window settings so "XMIN" is "0" and "XMAX" is "42." Let "YMIN" be "0" and "YMAX" be "5." All the data should be visible. You may also graph the data by hand.

4. What does the shape of the graph tell you about the motion of the plane and its acceleration?

5. Sketch what you think the velocity-time graph would look like.

6. Sketch what you think the displacement-time graph would look like.

Because a is equal to $\frac{\Delta v}{\Delta t}$, and we know the acceleration and time, we could rearrange the equation to find velocity, $\Delta v = a \cdot \Delta t$.

Because the time is in 1 second intervals, Δv is equal to a, which gives us the change in velocity from 1 second to the next. To find the actual velocity of the plane at each point, the consecutive changes in velocity must be added together to get the total velocity. The cumSum (cumulative sum) button on the calculator can do this for you. Go to Column C and label it "vel" for velocity. Go to the equation column and type in "=," then press the catalog button #1, scroll to cumulativeSum(and click "Enter." Next type in "B" to get the cumulative sum of Column B, which is the total change in acceleration. Create a scatterplot of a velocity versus time for the takeoff and see if it matches the sketch you drew in Problem 5. To do this, go to the "Data and Statistics" page. Scroll to the x-axis, go to "time," and click "Enter." Scroll to the y-axis, go to "vel," and click "Enter."

7. What does this graph tell you about the motion of the plane in terms of velocity?

Because v is equal to $\frac{\Delta d}{\Delta t}$, and we know the velocity and time, we could rearrange the equation to find displacement, $\Delta d = v \cdot \Delta t$.

Because the time is in 1 second intervals, Δd is equal to v, which gives us the change in displacement from 1 second to the next. To find the actual displacement of the plane at each point, the consecutive changes in displacement must be added together to get the total displacement. Again, use the cumSum (cumulative sum) feature on the calculator to do this. Go to Column D and label it "dis" for displacement. Go to the equation column and type in "=," then press the catalog button #1, scroll to cumulativeSum(and click "Enter." Next type in "C" to get the cumulative sum of Column C, which is the total change in velocity. Go to the "Data and Stats" page and create a scatterplot of displacement versus time. The x-axis is time. The y-axis is displacement. Does this match your sketch for Problem 6?

8. What does this graph tell you about the motion of the plane in terms of displacement?

Note: If you do not want to use the calculator, the graphs can be drawn by hand and the sums of the Columns may be added together by hand. You should get the same results. However, it will take quite a bit more time.

Section 6.14 Module Review

For Problem 1-6, use the properties of limits to evaluate the limits.

1. $\lim_{x \to 4} x^2$

2. $\lim_{x \to 4} (x^3 + 1)$

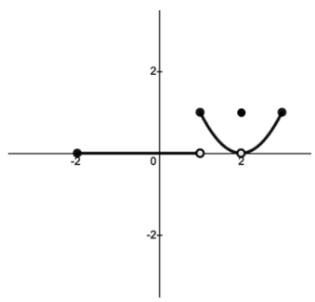
3. $\lim_{x \to -2} 5x^2$

4. $\lim_{x \to -1} (2x - 5)$

5. $\lim_{x \to 1} (x^3 + 2x^2 - 1)$

6. $\lim_{x \to 2} \frac{x^3+3x^2-2}{x^2+3}$

For Problem 7-15, evaluate the limits given the graph of the function $y = f(x)$.



7. $\lim_{x \to 3^+} f(x)$

8. $\lim_{x \to 3^-} f(x)$

9. $\lim_{x \to 2^+} f(x)$

10. $\lim_{x \to 2^-} f(x)$

11. $\lim_{x \to 1^+} f(x)$

12. $\lim_{x \to 1^-} f(x)$

13. $\lim_{x \to -2^+} f(x)$

14. $\lim_{x \to -2^-} f(x)$

15. Is the function $y = f(x)$ continuous? Explain why or why not?

For Problem 16-20, use the information given to solve the problem.

16. Is $g(x) = \sin x$ continuous over the domain that is a set of all real numbers?

17. Find the removable discontinuity for the function $f(x) = \frac{x^2+x-6}{x^2+2x-3}$.

18. Find the limit of the function as x approaches infinity:

$$f(x) = \frac{x+2}{x-5}$$

19. Sketch the distance function for the following situation:

Gabrielle leaves her house and drives to the store. It takes her 5 minutes to drive out of her neighborhood at a slow constant rate. She stops at a light for 2 minutes. For the next 7 minutes, she drives at a faster constant rate. She remembers that she left her purse at home, so she turns around and drives at a faster constant rate that gets her back to her neighborhood in 5 minutes. She slows down for the next 5 minutes until she gets to her home.

20. Sketch the velocity graph for Gabrielle driving toward the store and back.

Section 6.15 Module Test

For Problem 1-6, use the properties of limits to evaluate the limits.

1. $\lim_{x \to 4} x^3$

2. $\lim_{x \to 0} (x^2 - 5)$

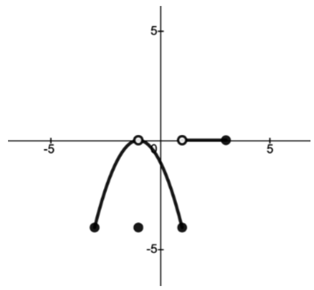
3. $\lim_{x \to -1} -3x^2$

4. $\lim_{x \to -2} (4x + 7)$

5. $\lim_{x \to 2} (x^3 - x^2 + 4)$

6. $\lim_{x \to 1} \frac{3x^3 - 2x^2 + 1}{x^2 + 1}$

For Problem 7-15, evaluate the limits given the graph of the function $y = f(x)$ below.



7. $\lim_{x \to 3^+} f(x)$

8. $\lim_{x \to 3^-} f(x)$

9. $\lim_{x \to 1^+} f(x)$

10. $\lim_{x \to 1^-} f(x)$

11. $\lim_{x \to -1^+} f(x)$

12. $\lim_{x \to -1^-} f(x)$

13. $\lim_{x \to -3^+} f(x)$

14. $\lim_{x \to -3^-} f(x)$

15. Is the function $y = f(x)$ continuous? Explain why or why not.

For Problem 16-20, use the given information to solve the problem.

16. Is the function $g(x) = \tan x$ continuous over the set of all real numbers?

17. Find the removable discontinuity for the function $f(x) = \frac{x^2+x-20}{x^2-2x-8}$.

18. Find the limit of the following function as x approaches infinity:

$$f(x) = \frac{-x}{5x-9}$$

19. Sketch the distance graph for the following situation:

A snail slowly crawls up a branch for 80 seconds at a steady constant pace. Then he sits and chews on a leaf for 60 seconds. Next, he continues to crawl up the branch at the same rate for 20 seconds and then rests for 20 seconds. Lastly, he crawls up to the end of the branch at the same pace for the next 15 seconds.

20. Sketch the velocity graph for the snail crawling up the branch in Problem 19.