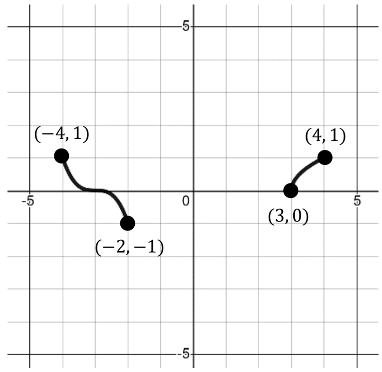
Pre-Calculus and Calculus Module 5 Limits and Continuity

Section 5.1 Piecewise Functions and Limits

Practice Problems 5.1

For Problem 1, tell which is the correct piecewise function for the graph (which is from Example 3).

1.



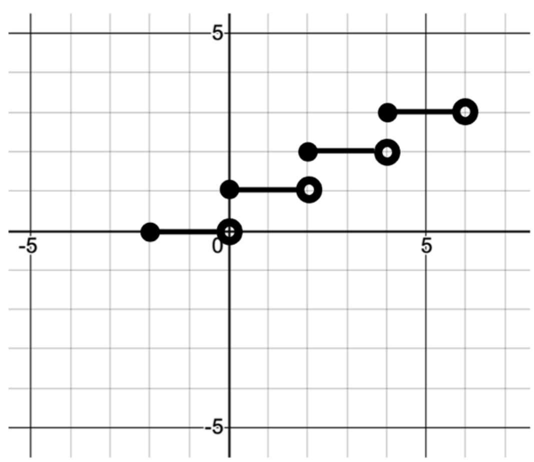
a) $f(x) = \begin{Bmatrix} (x+3)^3, -4 \le x \le -2 \\ \sqrt{x+3}, 3 \le x \le 4 \end{Bmatrix}$

b) $g(x) = \begin{Bmatrix} -(x+3)^3, -4 < x < -2 \\ \sqrt{x-3}, 3 < x < 4 \end{Bmatrix}$

c) $h(x) = \begin{Bmatrix} -(x+3)^3, -4 \le x \le -2 \\ \sqrt{x-3},\ 3 \le x \le 4 \end{Bmatrix}$

d) $j(x) = \begin{Bmatrix} -(x-3)^3, -4 \le x \le -2 \\ \sqrt{x+3}, 3 \le x \le 4 \end{Bmatrix}$

For Problem 2-6, use the piecewise function below to solve the problem.



2. Write the piecewise function with intervals for the graph.

3. Is $f(x) = 1.5$ a solution of the piecewise function?

4. What is x when $f(x) = 3$?

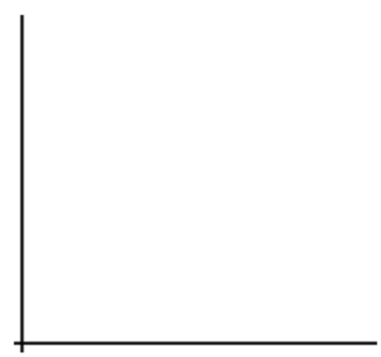
5. What is another name for the piecewise function?

6. Is the function continuous or discontinuous?

For Problem 7-9, use the graph you make in Problem 7 to solve the problem.

7. Draw the graph of the piecewise function described: The number of children in an after-school program is given below. Let the x-axis be the age of the children and let the y-axis be the number of children.

Age of children	Number of Children
$[5, 6)$	50
$[6, 7)$	100
$[7, 8)$	120
$[8, 9)$	80
$[9, 10)$	50

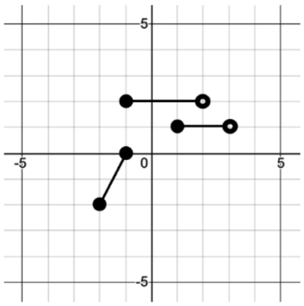


8. Why does the step function have closed intervals on the left and open intervals on the right?

9. Is the step function continuous or discontinuous?

For Problem 10-12, use the information given to solve the problem.

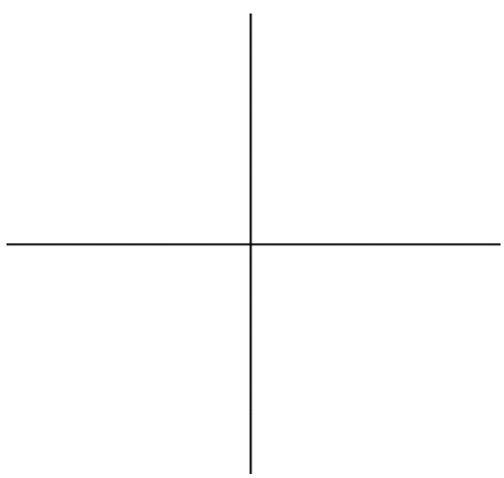
10. Is the graph below a piecewise function? Why or why not?



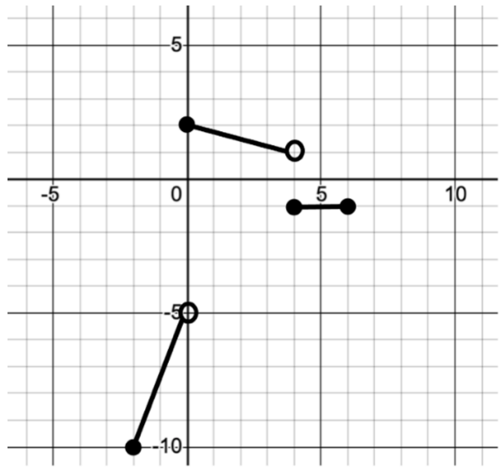
11. Is $\mathrm{m}(x)$ a piecewise function? Why or why not?

$$\mathrm{m}(x) = \begin{cases} -2, -4 \le x \le -2 \\ x, -2 < x \le 1 \\ \dfrac{2}{3}x + 1, 1 < x \le 3 \end{cases}$$

12. Draw the graph of $\mathrm{m}(x)$ in Problem 5. Write the intervals using interval notation.



For Problem 13-20, use the figure of $g(x)$below to solve the problem.



13. Which piece of the graph depicts $g(x)$ moving closer and closer to 0 from the left side? When x gets close to 0 from the left side, what does $g(x)$ (the y-value) get close to?

14. Which piece of the graph depicts $g(x)$ moving closer and closer to 0 from the right side? When x is 0, what does $g(x)$ (the y-value) become?

15. Is there one value that $g(x)$ (the y-value) gets close to when x gets close to 4 from both sides? Do you think a limit exists at $x = 4$?

16. What is the exact value of $g(4)$?

17. What type of functions are depicted by each of the three pieces of the graph of $g(x)$?

18. Write the function for $g(x)$ over the interval $-2 \leq x < 0$.

19. Write the function for $g(x)$ over the interval $0 \leq x < 4$.

20. Write the function for $g(x)$ over the interval $4 \leq x \leq 6$.

Section 5.2 Rational Functions and Limits

Practice Problems 5.2

For Problem 1-6, tell whether or not the function is a rational function and why it is or is not.

1. $f(x) = \frac{3+x}{x-2}$

2. $g(x) = \frac{3-2x-5x^2}{2x^2+4}$

3. $h(x) = \frac{\sqrt[3]{x^2}}{2x^2+3x-5}$

4. $f(x) = x^3 - 4$

5. $f(x) = \sqrt[4]{y}$

6. $f(x) = -\frac{1}{(x+4)^2}$

For Problem 7-10, use the information below to solve the problem.

The $\lim\limits_{x \to c} f(x) =$ L means the limit as x approaches c is L, given real numbers c and L. Moreover, this means that L is the one number that $f(x)$ gets arbitrarily close to as x gets closer to c on both sides, but not equal to c. This is read: "The limit of f as x approaches c is L." There are also limits involving infinity as x usually read as 'The limit of f as x approaches infinity is L (as x becomes arbitrarily large).

Go back to Example 1 and match the correct explanation in a)-d) with the limit statement for which the parent rational function is $f(x) = \frac{1}{x}$.

7. $\lim_{x \to 0^-} \left(\frac{1}{x}\right) = -\infty$ (This means as x approaches 0 from the left side, or negative side)

8. $\lim_{x \to 0^+} \left(\frac{1}{x}\right) = \infty$ (This means as x approaches 0 from the right side, or positive side)

9. $\lim_{x \to \infty} \left(\frac{1}{x}\right) = 0^{(+)}$

10. $\lim_{x \to -\infty} \left(\frac{1}{x}\right) = 0^{(-)}$

For Problem 11-13, use the information given to solve the problem.

11. If we look at the table below when $f(x) = \frac{1}{x}$, we see that as x gets bigger and bigger moving towards positive infinity, $f(x)$ gets closer and closer to 0. Write the limit statement for this table. Match the limit statement with the explanation in a)-d) from Example 1 like we did in Problem 7-10.

x	$f(x)$
100	0.01
1,000	0.001
10,000	0.0001

12. Use the table below to complete the following statement when $f(x) = \frac{1}{x}$:

x	$f(x)$
-100	-0.01
$-1{,}000$	-0.001
$-10{,}000$	-0.0001

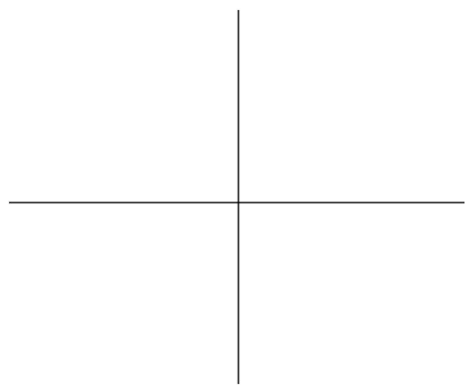
As x decreases without bound toward ____________________ infinity, the function $\frac{1}{x}$

approaches ____________________ from the negative side.

13. Problems 11 and 12 represent the end-behavior of the function $y = \frac{1}{x}$ or $f(x) = \frac{1}{x}$. Problem 11 describes the end behavior of the right side of the rational function $f(x) = \frac{1}{x}$. Problem 12 describes the end behavior of the left side of the rational function $f(x) = \frac{1}{x}$. The limit in Problem 12 could be written $\lim_{x \to -\infty} f(x) = 0$ or $\lim_{x \to -\infty} (\frac{1}{x}) = 0$. Why is either $\lim_{x \to \infty} f(x) = 0$ or $\lim_{x \to \infty} (\frac{1}{x}) = 0$ acceptable for Problem 11?

For Problem 14-20, use the graph in Problem 14 and the table in Problem 15 to solve the problem.

14. Draw the graph of $f(x) = -\frac{1}{x}$.



15. Complete the table for $f(x) = -\frac{1}{x}$.

x	$f(x)$
-1	
-0.1	
-0.01	

16. Fill in blanks: As x _______________ towards 0 from the left, $f(x)$ _______________ towards ___________________ ____________________.

17. Complete the limit, read "The limit of *f* as *x* approaches 0 from the left:"

$$\lim_{x \to 0^-} f(x) = \_\_\_\_\_\_\_\_\_\_$$

18. Complete the table for $f(x) = -\frac{1}{x}$.

x	$f(x)$
1	
0.1	
0.01	

19. Fill in the blanks: As x _______________ towards 0 from the right, $f(x)$ _______________ towards _______________ ________________.

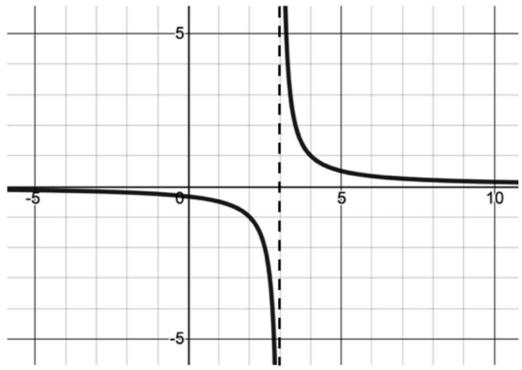
20. Complete the limit, read "The limit of *f* as *x* approaches 0 from the right:"

$$\lim_{x \to 0^+} f(x) = \_\_\_\_\_\_\_\_\_\_$$

Section 5.3 One-sided and Two-sided Limits

Practice Problems 5.3

For Problem 1-4, use the given graph of $h(x) = \frac{1}{x-3}$ to solve the problem.



1. The function $h(x)$ has a vertical asymptote at $x =$ ___________.

2. Find $\lim_{x\to 3^-} h(x)$.

3. Find $\lim_{x\to 3^+} h(x)$.

4. Find $\lim_{x\to 3} h(x)$.

5. Find $\lim_{x\to\infty} (\sin x)$.

6. Find $\lim_{x\to\infty} 3\sin x$.

For Problem 7-10, use the information given to solve the problem.

7. Let $f(x) = \frac{2x-1}{x+3}$. Find $f(10)$, $f(100)$, $f(1{,}000)$. As x increases, what happens to the values of $(f(x))$?

8. The graphing function for the parent function of a rational equation is $y = \frac{a}{x-b} + \text{c}$ where b is the horizontal shift and c is the vertical shift. Given the equation $y = \frac{2}{x-4} + 1$, what is the vertical asymptote? What is the horizontal asymptote?

9. a) Find $\lim_{x\to 4^-} \frac{2}{x-4} + 1$. b) Find $\lim_{x\to 4^+} \frac{2}{x-4} + 1$.

10. Simplify the rational expression:

$$\frac{\frac{-2x}{y}+4}{\frac{2x^2}{y^2}+2}$$

For Problem 11-14, find the limits of the function given.

11. $\lim_{x\to\infty} g(x)$ when $g(x) = \frac{100}{x}$

12. $\lim_{x\to\infty} f(x)$ when $f(x) = 10x^3 - 100x^2$

13. $\lim_{x\to\infty} h(x)$ when $h(x) = (\frac{1}{2})^x$

14. $\lim_{x\to\infty} m(x)$ when $m(x) = 5\sqrt{x}$

For Problem 15-20, use the information given to solve the problem.

15. Substitute large values of x and graph the function in a window with large x-values to find $\lim n(x)$ as $x \to \infty$ if $n(x) = \frac{4x^3+2x}{x^3-100}$.

16. Sketch the graph of $f(x) = \frac{1}{x+1} - 2$.

17. Sketch the graph of $f(x) = \frac{1}{x-3} + 4$.

18. Factor $f(x) = \frac{3x-2}{x-2}$ as a transformation of $g(x) = \frac{1}{x}$ and sketch the graph of $f(x)$.

19. Factor $f(x) = \frac{2x-1}{x+1}$ as a transformation of $g(x) = \frac{1}{x}$ and sketch the graph of $f(x)$.

20. If $f(x)$ varies directly as $(x + 1)$ and $f(3) = 3$, find $f(-1)$. Use $f(x) = k(x + 1)$ and find k (the constant of variation) first.

Section 5.4 Finding Limits as x Approaches a Constant

Practice Problems 5.4

For Problem 1-5, calculate the limits.

1. $\lim_{x \to 2} \frac{x^2-4}{x-2}$

2. $\lim_{x \to 3} 5x^2$

3. $\lim_{x \to -2} \frac{3x^3}{2x+7}$

4. $\lim_{x \to 2} \frac{x^2-4}{x^4}$

5. $\lim_{x \to 9} \frac{3-\sqrt{x}}{x-9}$

For Problem 6-20, use the information given to solve the problem.

6. Is $\lim_{x\to 2^+} \frac{x^2}{2}$ a one-sided limit or a two-sided limit?

7. Fill in the blanks:

$$\lim_{x\to 0} \frac{\sin(2x)}{7x} = \frac{\sin 2\cdot 0}{7\cdot 0} = \frac{\sin \_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_} = \frac{\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_}$$

This is called ____________________.

8. Find the limit:

$$\lim_{x\to 0} \frac{\cos 2x - 1}{\cos^2 x - 1}$$

Use the trigonometric identity $\cos(2x) = 2\cos^2 x - 1$ if the above is indeterminate.

9. If $\lim_{x\to 2^-} f(x) = 4$ and $\lim_{x\to 2^+} f(x) = \frac{1}{2}$, what does that tell you about the limit?

10. Use the function $f(x) = \left\{\begin{matrix} a + bx, x > 1 \\ 2, x = 1 \\ -ax^2 + b, x < 1 \end{matrix}\right\}$ to find the values of a and b so that $\lim_{x \to 1} f(x)$ exists.

11. To estimate what the $\lim_{x \to 2^+} f(x)$ approaches when $f(x) = 10 - x^3$ try at least three integer values to the right of 2 approaching 2 for x Find $f(5), f(4),$ and $f(3)$.

12. To estimate what the $\lim_{x \to 2^-} f(x)$ approaches when $f(x) = 10 - x^3$ try at least three integer values to the left of 2 approaching 2 for x. Find $f(-1), f(0),$ and $f(1)$.

13. Find $\lim_{x \to 2} f(x)$ when $f(x) = 10 - x^3$.

14. Let $f(x)$ be equal to $10 - x^3$. Does $f(x)$ seem to be approaching 2 from the left? Does $f(x)$ seem to be approaching 2 from the right?

15. Verify $\lim_{x \to 2} f(x)$ is 2 when $f(x)$ is $10 - x^3$ using a graphing calculator.

16. Find $\lim_{x \to 3} 3x^2$ by substitution. Find $\lim_{x \to 3} 3x^2$ using the properties from the Lesson Notes.

17. Demonstrate that the $\lim_{x \to 5} \frac{x+5}{x^2-25}$ is undefined?

18. Factor the denominator and find $\lim_{x \to 5} \frac{x+5}{x^2-25}$. Is it indeterminate or undefined?

19. Sketch the graph of $f(x) = \frac{x+5}{x^2-25}$. What are the asymptotes?

20. Is $\lim_{x \to 5} \frac{x^2-25}{x+5}$ indeterminate, zero, or undefined? Factor the numerator first. Sketch a graph of the function and label the point of discontinuity.

Section 5.5 Finding Limits as x Approaches Infinity

Practice Problems 5.5

For Problem 1-6, find the limit of the rational function as x approaches infinity.

1. $f(x) = x^2 - 2$

2. $f(x) = -3x^3 + 4$

3. $f(x) = \frac{2x+5}{2x+3}$

4. $f(x) = \frac{4x^3-3}{-2x^2+5x}$

5. $f(x) = \frac{3x^2+2x-8}{5x^4-2x^2+1}$

6. $f(x) = \frac{-5x^3}{x^2-1}$

For Problem 7-11, use the function $f(x) = -\frac{3}{x-2} + 4$ to solve the problem.

7. Find $\lim_{x \to \infty} -\frac{3}{x-2}$.

8. Find $\lim_{x \to \infty} 4$.

9. Find $\lim_{x \to \infty} -\frac{3}{x-2} + 4$.

10. The equation is of the form of the general equation of a rational function: $y = \frac{a}{x+h} + k$. Is the limit of $f(x)$ equal to the value of a, h, or k as x approaches infinity?

11. What does k represent on the graph of the function?

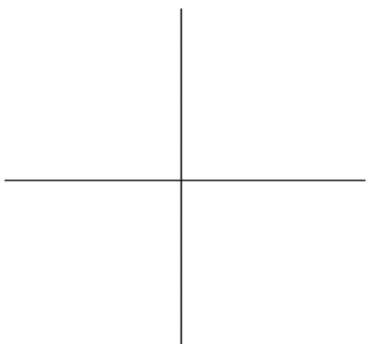
For Problem 12-16, let $a(x) = \frac{3^x}{x^4}$ and $b(x) = \log(a(x))$.

12. Simplify $b(x)$ by substituting $a(x)$ into the equation and using the rules of exponents.

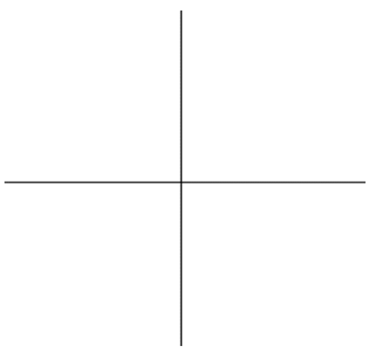
13. Find $\lim_{x \to \infty} a(x)$ and explain your reasoning.

14. Find $\lim_{x \to \infty} b(x)$ and explain your reasoning.

15. Use a graphing calculator to sketch the graph of $a(x)$ and verify your solution to Problem 13. Where is there an asymptote and why is there an asymptote?



16. Use a graphing calculator to sketch the graph of $b(x)$ and verify your solution to Problem 14. Where is there an asymptote and why is there an asymptote?

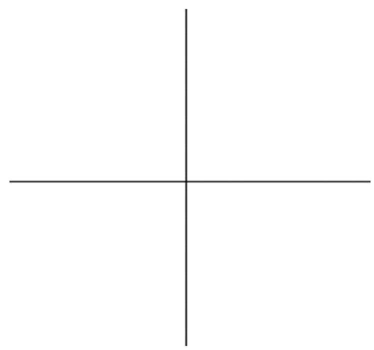


For Problem 17-20, use the information given to solve the problem.

17. Write the equation of a rational function that has a horizontal asymptote at $y = 10$ and a vertical asymptote at $x = -7$ where $a = 1$.

18. Write the equation of a rational function that has a horizontal asymptote at $y = -9$ and a vertical asymptote at $x = 5$ where $a = 1$.

19. Sketch the graph of $f(x) = \frac{1}{x+5} - 3$. Use this graph to evaluate the limit statements given below.



a) $\lim_{x\to\infty} f(x)$ (read: "the limit of f of x as x approaches positive infinity.")

b) $\lim_{x\to-5^-} f(x)$ (read: "the limit of f of x as x approaches negative five from the left side.")

c) $\lim_{x\to-5^+} f(x)$ (read: "the limit of f of x as x approaches negative five from the right side.")

20. Find the limits of the function $f(x) = -11$.

a) $\lim_{x\to+\infty} f(x)$

b) $\lim_{x\to-\infty} f(x)$

Section 5.6 Asymptotes and Limits

Practice Problems 5.6

For Problem 1-4, fill in the blanks to complete the steps for finding vertical and horizontal asymptotes of rational functions.

1. Set the denominator equal to 0 and solve for x. The ____________________ asymptotes are not included in the domain.

2. Compare the degree of the numerator to the degree of the denominator. If they are the same, then the ____________________ asymptote is the ratio of the lead coefficients.

3. If the degree of the denominator is greater than the degree of the ____________________ then there is a horizontal asymptote at $y = 0$.

4. If the degree of the numerator is greater than the degree of the ____________________, use long division or synthetic division to find the slant oblique asymptote.

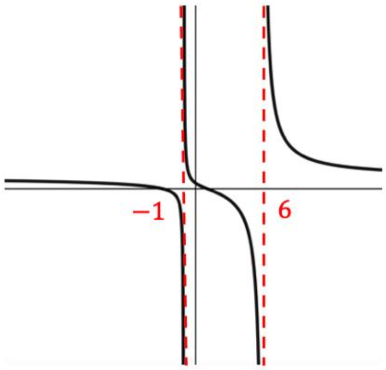
Note: There are exceptions to these rules if something in the numerator cancels with something in the denominator.

For Problem 5-9, use $f(x) = \frac{x^2+2x-3}{x^2-5x-6}$ to solve the problem.

5. Factor the denominator and set the factors equal to 0 to solve for x. What is the domain of the function?

6. What are the vertical asymptotes of the function?

7. From the graph below, what can you say about the end behavior of the function as $|x| \to 0$? Investigate $x \to -\infty$ when $x < -1$ and $x \to \infty$ when $x > 6$. Because $(x+1)$ factors out, the graph will never look like this. The vertical asymptote at $x = -1$ disappears and becomes a point of discontinuity at $(-1, 0)$ where x becomes a shared root. This is a limitation in using a graphing utility.



8. Write four approach statements as $x \to -1^-$ and $x \to -1^+$, and $x \to 6^+$ and $x \to 6^-$.

9. When $x < -1$, there appears to be an asymptote at $y = 1$; when $x > 6$, there appears to be an asymptote at $y = 1$. Why do you think this happens?

10. Write $f(x) = \frac{x^2-2x-3}{x^2-5x-6}$ in its factored form and simplify it.

11. Find the a) vertical asymptote and b) horizontal asymptote of the simplified form of $f(x)$ in Problem 10 and any holes or discontinuities.

For Problem 12-16, find the asymptotes.

12. Find the a) vertical asymptote and b) horizontal asymptote for $f(x) = \frac{2x+3}{x^2-1}$.

13. Find the a) vertical asymptote and b) horizontal asymptote for $f(x) = \frac{x^3-2}{x}$.

14. Find the a) vertical asymptote and b) horizontal asymptote for $f(x) = \frac{x^2+2x-3}{x^2+5x+6}$. Are there any holes or discontinuities?

15. Find the a) vertical asymptote and b) horizontal asymptote for $f(x) = \frac{2x^3+5x}{3x+1}$.

16. Find the a) vertical asymptote, b) horizontal asymptote, and c) oblique asymptote(s) for $f(x) = \frac{3-x^2}{x+2}$. Use a graphing calculator to check your asymptotes.

For Problem 17-20, use the information given to solve the problem.

17. If $\lim_{x \to \pm\infty} f(x) = a$, what is the horizontal asymptote of the function?

18. If $\lim_{x \to b} f(x) = \pm\infty$, what is the vertical asymptote of the function?

19. Verbally describe $\lim_{x \to \pm\infty} f(x) =$ L.

20. Verbally describe $\lim_{x \to \mathrm{c}} f(x) = \pm\infty$.

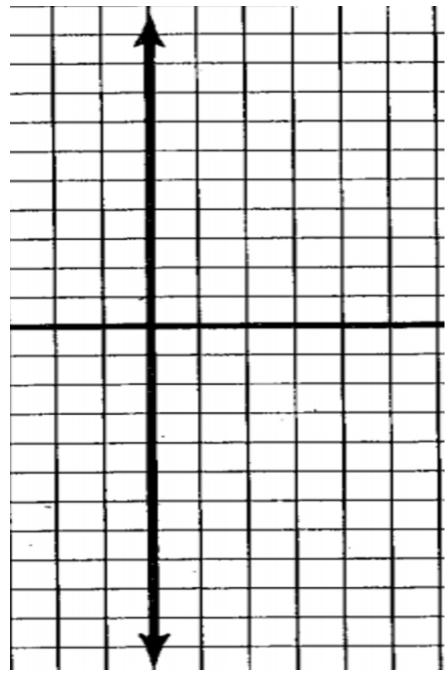
Section 5.7 Continuity and Discontinuity

Practice Problems 5.7

For Problem 1-5, use the piecewise function below to solve the problem.

$$f(x) = \begin{Bmatrix} x^3 + 1, x \leq -1 \\ x^2 + 2, -1 < x \leq 3 \\ x + 2, x > 3 \end{Bmatrix}$$

1. Check to see if the piecewise function is continuous or discontinuous when $x = 3$.



2. Find $f(2)$ using the piecewise function in Problem 1.

3. Draw the graph of the piecewise function in Problem 1.

4. Find $f(-5)$ using the piecewise function.

5. If a polynomial function is continuous on its domain, is $f(x)$ equal to $x + 2$ continuous when x is equal to 7?

For Problem 6-10, use the given piecewise function and solve the problems to answer the following question.

$$f(x) = \begin{Bmatrix} x^2 + 5, x < 1 \\ ax + 9, x \geq 1 \end{Bmatrix}$$

For what value of a is the piecewise function continuous over the entire set of real numbers?

6. Which equation would you use to find $\lim_{x \to 1^-} f(x)$?

7. Find $\lim_{x \to 1^-} f(x)$.

8. Which equation would you use to find $\lim_{x \to 1^+} f(x)$?

9. Find $\lim_{x \to 1^+} f(x)$ in terms of a.

10. For the graph to be continuous the limits must be the same as they approach 1 from the left and right. Set the limits equal and solve for a to answer the question before Problem 6?

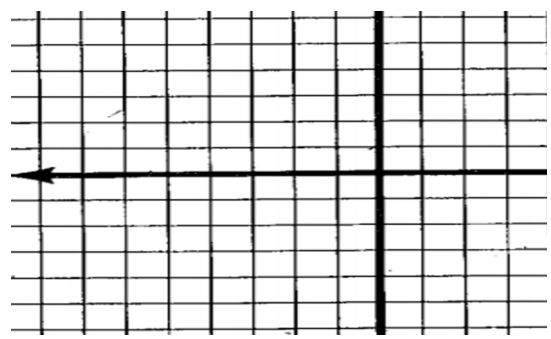
For Problem 11-15, use the given function to solve the problem.

$$f(x) = \frac{x^2 + x - 2}{x^2 + 2x - 3}$$

11. Is the graph of the function continuous?

12. Find the vertical and horizontal asymptotes and holes in the function.

13. Sketch the graph of the function.



14. How would changing the function to $f(x) = \frac{x^2+x-2}{x^2+2x-3} + 4$ change the graph of the function?

15. Find the x-intercept and y-intercept of the function in Problem 14? How can you verify whether the graph is continuous or discontinuous?

For Problem 16-20, tell whether the function given for $y = f(x)$ is continuous or discontinuous.

16. a) The reciprocal function $y = \frac{1}{x}$ over its domain

b) The reciprocal function $y = \frac{1}{x}$ over the set of all real numbers

17. Polynomial Functions

18. Trigonometric Functions ($y = \cos x$ and $y = \sin x$)

19. A) Rational functions over their domain

B) Rational functions over the set of real numbers

20. The greatest integer function $y = [x]$

Note: The greatest integer function may also be written using the notation $y = \lceil x \rceil$ for the ceiling, and the least integer function may also be written using the notation $y = \lfloor x \rfloor$ for the floor.

Section 5.8 Definition of Continuity

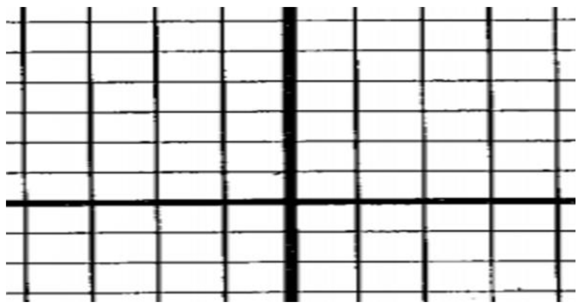
Practice Problems 5.8

For Problem 1-10, use $h(x)$ given below to solve the problem.

For Problem 1-5, find the limits of the piecewise function over the given interval.

$$h(x) = \left\{ \begin{array}{c} \frac{1}{x^2}, x < -2 \\ 3, -2 \leq x < 2 \\ 5, x = 2 \\ x + 1, 2 < x \leq 3 \end{array} \right\}$$

First, draw the graph of the piecewise function:



1. $\lim_{x \to -2^-} h(x)$

2. $\lim_{x \to -2^+} h(x)$

3. $\lim_{x \to 0} h(x)$

4. $\lim_{x \to 2^+} h(x)$

5. $\lim_{x \to 2^-} h(x)$

6. Is $h(x)$ continuous or discontinuous? Explain why or why not.

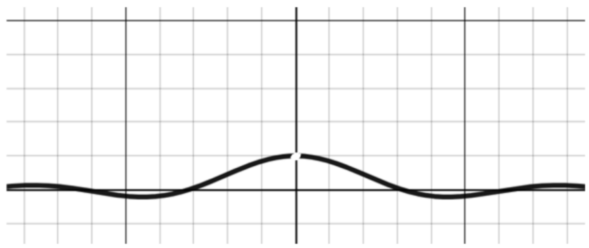
7. Is $\lim_{x \to -2} h(x)$ equal to $h(-2)$?

8. Given the piecewise function, are the three conditions of continuity met for $h(0)$?

9. Which of the three conditions for the piecewise function are not met for $h(2)$?

10. Where is another discontinuity in the piecewise function where these conditions of continuity are not met?

For Problem 11-13, given the function $f(x) = \frac{\sin x}{x}$ solve the problem.



$\boldsymbol{x}$	$\boldsymbol{f(x)}$
$-\pi$	0
$-\frac{\pi}{2}$	$\frac{2}{\pi} \approx 0.6366$
$-\frac{\pi}{4}$	$\sin 1 \approx 0.84$
0	Indeterminate
$\frac{\pi}{4}$	0.84
$\frac{\pi}{2}$	0.636
π	0

11. Does the graph have any x-intercepts?

12. Does the graph have a horizontal asymptote?

13. What happens to $f(x)$ as $x \to \infty$? Find $\lim_{x\to\infty} \frac{\sin x}{x}$.

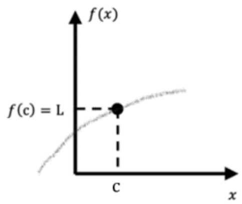
For Problem 14 and 15, given the function $f(x) = \frac{x^2-4}{x-2}$ solve the problem.

14. Is the function continuous at $x = 2$?

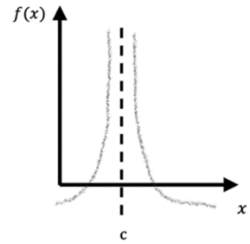
15. Does the limit of the function exist as $x \to 2$? Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

For Problem 16-19, match the graph with the explanation.

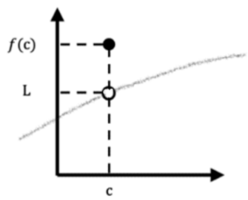
a)



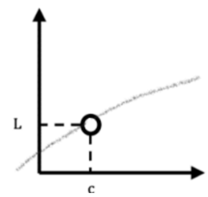
b)



c)



d)



16. The graph is not continuous. There is no $f(\mathrm{c})$. There is a removable discontinuity.

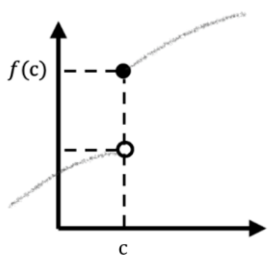
17. The graph is not continuous, and $f(\mathrm{c}) \neq \mathrm{L}$.

18. There is a vertical asymptote. This graph has infinite discontinuity.

19. The graph is continuous and $f(\mathrm{c}) = \mathrm{L}$.

For Problem 20, use the information given to solve the problem.

20. What type of discontinuity does the graph below have? Is it removable?



Section 5.9 Logarithms and Limits

Practice Problems 5.9

For Problem 1, use Example 3 from the Lesson Notes to solve the problem.

1. Find $\lim_{x\to\infty} \log_b x$. Let $b = 1.5$.

For Problem 2-5, think about the graph of the function given and find its limits.

2. $\lim_{x\to\infty} g(x)$ if $g(x) = 5x^4$

3. $\lim_{x\to\infty} h(x)$ if $h(x) = 3\sqrt{x}$

4. $\lim_{x\to\infty} m(x)$ if $m(x) = (\frac{1}{3})^x$

5. $\lim_{x\to\infty} f(x)$ if $f(x) = 3.5^x$

For Problem 6-9, use the given information to solve the problem.

Below are five functions that go to infinity as x approaches infinity.

1.5^x $\quad$ $3\sqrt{x}$ $\quad$ $2\log x$ $\quad$ $5x^4$ $\quad$ $3^x - 10$

6. Find the order from the slowest to reach infinity to the fastest to reach infinity.

7. Find the value of each of the five functions when $x = 0$. What values of the function are these if defined?

8. Find the value of each of the five functions when $x = 1$, $x = 10$, $x = 100$, and $x = 1{,}000$.

x	$3\sqrt{x}$	$5x^4$	1.5^x	$2\log x$
1				
10				
100				
1,000				

Does the table confirm your order in Problem 6?

9. Sketch the graphs of each of the five functions. Do the graphs appear to confirm the order in Problem 6?

For Problem 10, solve the multiple-choice problem.

10. Which of the functions does not reach infinity as x approaches infinity. Why?

a) $g(x) = -10x^4$

b) $h(x) = \frac{5x^5+3x}{x^2-100}$

c) $m(x) = 5x^7 - 2x$

d) $n(x) = 20\log_4 x$

For Problem 11-14, solve for y in terms of x by finding the logarithmic form of each function.

11. $x = 1.5^y$

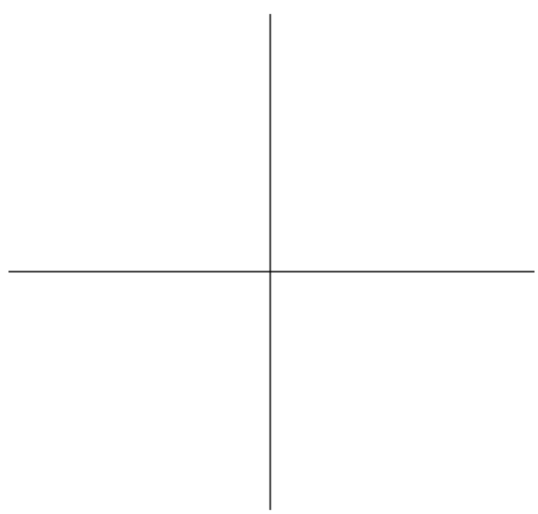
12. $x = 6^y$

13. $x = (\frac{1}{3})^y$

14. Find the inverse of each function in Problem 11-13 and write it in exponential form using function notation.

For Problem 15, sketch the graphs of Problem 11-13 in red and sketch the inverse of each in blue.

15.



For Problem 16-20, solve the logarithms for $f(2)$ and $f(0.3)$. Give the decimal approximation to the ten-thousandth place.

16. $f(x) = \log_{10} x$

17. $f(x) = \log_{4} x$

18. $f(x) = \log_{0.1} x$

19. What bases have the same value, but opposite signs for $f(2)$ and $f(0.3)$?

20. What other base would have the same value but opposite sign for the output of $f(2)$ and $f(0.3)$ given the function $f(x) = \log_{0.01} x$

Section 5.10 Limits and the Natural Logarithm

Practice Problems 5.10

For Problem 1-16, use the information given to solve the problem.

1. Write $\log_e 45$ using natural logarithm notation.

2. Write $\log_e 45 = y$ using exponential notation.

3. Use a check and guess method to estimate a value for y that makes $\log_e 45$ approximately true.

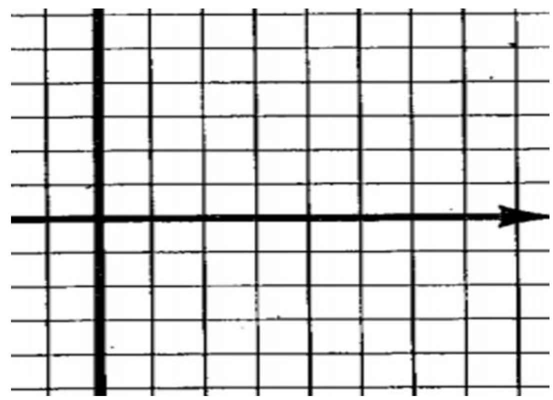
4. Use your calculator and natural logarithm key to find a better estimate for your solution in Problem 3.

5. Find $\lim\limits_{x\to 0^+} \log x$.

6. Why is it unnecessary to find $\lim\limits_{x\to 0^-} \log x$?

7. Find $\lim_{x \to 0^+} \ln x$.

8. Make a table of values for $f(x) = \ln x - 3$ and sketch a graph of the function.



9. Find $\lim_{x \to 0^+} [\ln(x) - 3]$ and explain what is happening and why.

10. Find $\lim_{x \to \infty} [\ln(x) - 3]$ and explain what is happening and why.

11. Complete the table for $f(x) = \frac{1}{e^x}$.

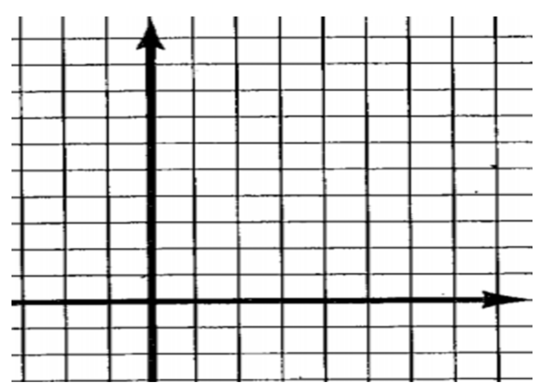
x	$f(x)$
-3	
-2	
-1	
0	
1	
2	
3	
4	

12. Based on the table from Problem 11, find $\lim_{x\to\infty} \frac{1}{e^x}$.

13. Find $\lim_{x\to 0^+} \frac{1}{e^x}$.

14. Find $\lim_{x\to 0^-} \frac{1}{e^x}$.

15. Draw the graph of $\frac{1}{e^x}$. Do there appear to be any asymptotes? What is the y-intercept?



16. Find $\lim_{x \to -\infty} \frac{1}{e^x}$.

For Problem 17-20, fill in the blanks and demonstrate the validity of the properties. Note that the properties of logarithms hold true for natural logarithms.

17. $\ln ax = \ln a +$ _______________

Demonstrate the validity of the properties when $a = 3$ and $x = 4$.

18. $\ln \frac{a}{x} =$ _______________ $- \ln x$

Demonstrate the validity of the properties when $a = 2$ and $x = 5$.

19. What is $\ln 1$? What is the property for $\ln \frac{a}{x}$ when $a = 1$?

20. Use the power rule:

$\ln x^n =$ _______________ $\ln$ _______________

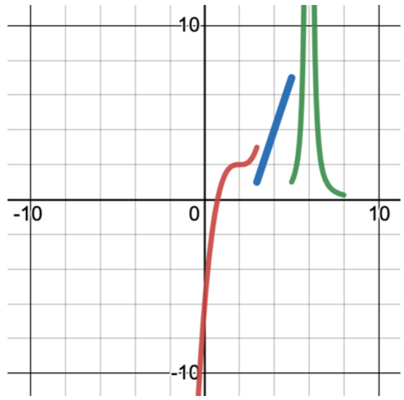
Demonstrate the validity of the properties when $n = -2$ and $x = 6$.

Section 5.11 Using Technology to Analyze Limits

Practice Problems 5.11

For Problem 1-11, given the graph of the function below solve the problem. Check your solution using technology.

$$f(x) = \begin{cases} (x-2)^3 + 2 \{-4 \le x \le 3\} \\ 3x - 8 \{3 < x < 5\} \\ \dfrac{1}{(x-6)^2} \{5 < x < 8\} \end{cases}$$



1. Find $f(3)$.

2. Find $\lim_{x \to 3^+} f(x)$.

3. Find $\lim_{x \to 3^-} f(x)$?

4. Where is the function undefined?

5. Is there an asymptote at $x = 3$?

6. Where does an asymptote exist?

7. Find $f(4)$.

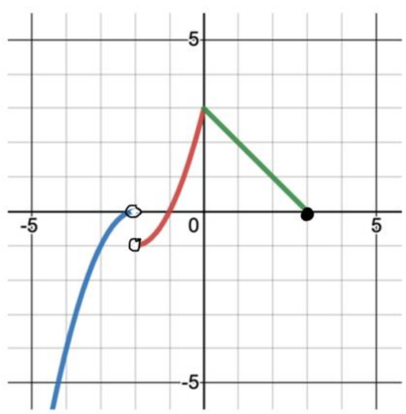
8. Find $f(5)$.

9. Find $f(8)$.

10. Find $f(7)$.

11. Find $f(-5)$.

For Problem 12-20, use the given graph to solve the problem.



12. Find $\lim_{x \to -2^-} f(x)$.

13. Find $\lim_{x \to -2^+} f(x)$.

14. Find $\lim_{x \to 2} f(x)$.

15. Find $\lim_{x \to 3} f(x)$.

16. Find $f(3)$.

17. Is there a hole or a jump at $x = -2$?

18. What is $\lim_{x \to -2} f(x)$? What is $f(-2)$?

19. Is $x = -2$ a removable or non-removable discontinuity?

20. The discontinuity at $x = 0$ is called a cusp because $f(c) =$ L. What is the limit of $f(x)$ as x approaches 0?

Section 5.12 Applications of Limits

Practice Problems 5.12

For Problem 1-3, use the information given to solve the problem.

1. Try $n = 10$, $n = 100$, and $n = 1{,}000$ to investigate $\lim_{n\to\infty}(1 + \frac{2}{n})^n$.

2. Find e^2 and $2e$? Which is closer to the limiting values found in Problem 1?

3. Which is more accurate?

$$\lim_{n\to\infty}(1 + \tfrac{2}{n})^n \approx e^2 \text{ or } \lim_{n\to\infty}(1 + \tfrac{2}{n})^n \approx 2e$$

For Problem 4 and 5, write the expression in terms of the irrational number e then approximate a solution.

4. $\lim_{n\to\infty}(1 + \frac{0.4}{n})^n$

5. $\lim_{n\to\infty}(1 + \frac{5}{n})^n$

For Problem 6-9, use the information given to solve the problem.

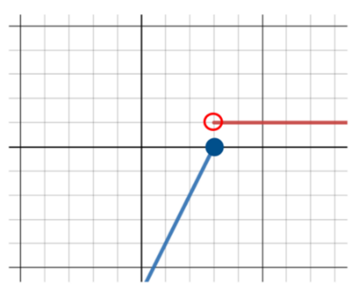
6. The Bronze Bank wants to have your account at the same annual interest rate, 5.75%, as Gold Bank and Silver Bank. They offer you the 5.75% rate compounded hourly; will you switch banks?

7. Missionaries in Brazil often buy building supplies for churches when the interest rates on bank loans are low. They will save the supplies for months and buy more when the rates dip again. Presently, the Central Bank rate in Brazil is 12.75%. If missionaries borrow $\$2{,}500.00$ for five years at this rate compounded continuously, how much will the missionaries owe the Central Bank after five years?

8. The Swiss National Bank has a rate of -0.75%. What does this mean for a loan that must be repaid in five years?

9. The Federal Reserve of the United States has an interest rate of 0.0250% compounded continuously. If $\$1{,}125.00$ is borrowed, how much interest accrues on it after three years?

For Problem 10-12, use the given graph to solve the problem.

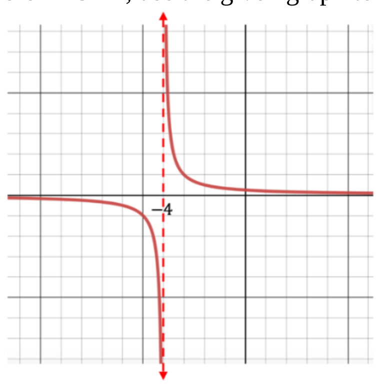


10. Explain why the limit does not exist as $x \to 3$.

11. Is there a removable discontinuity at $x = 3$?

12. What is $f(3)$?

For Problem 13-17, use the given graph to solve the problem.



13. Explain why the limit does not exist at $x \to -4$.

14. Is there a removable discontinuity at $x = 4$?

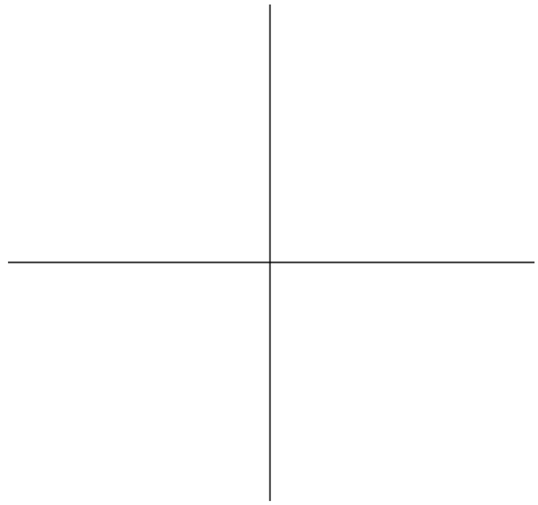
15. Is the function continuous over all real numbers?

16. What is the domain of the function?

17. Is the graph continuous over the given domain of the function?

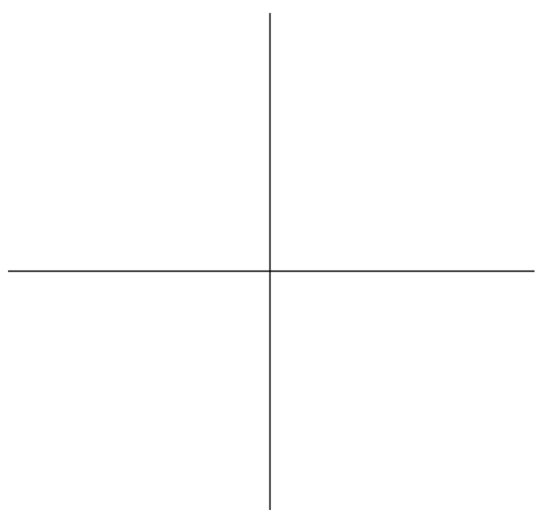
For Problem 18-20, use the information given to solve the problem.

18. What is $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$? Solve the problem algebraically and confirm it graphically.



19. Is $f(x) = \frac{x^2+x-6}{x-2}$ continuous or discontinuous? If it is discontinuous, is it a removable discontinuity?

20. Evaluate $\lim_{x \to 0} \frac{|x|}{x}$. Sketch its graph.

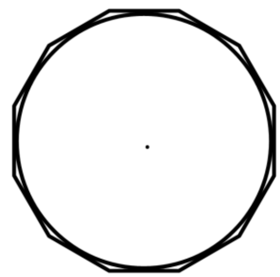


<u>Section 5.13 Archimedes and Approximating Pi</u>

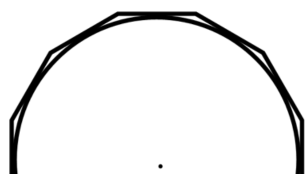
<u>Practice Problems 5.13</u>

For Problem 1 and 2, use the information given to solve the problem.

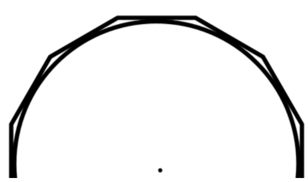
1. Follow the steps to find the area of the dodecagon circumscribed about the unit circle.



a) What is the central angle of each triangle?



b) What is the degree of each half of the central angle when bisected by the height of each isosceles triangle?



c) Use trigonometry (Hint: try tangent) to find the base of each isosceles triangle.

(Hint: The side opposite the 15° angle is only half the base of the isosceles triangle, and the height of the triangle is the radius of the unit circle.)

d) Use the answer from c) to find the area of the isosceles triangle.

e) Use the answer from d) to find the area of the circumscribed dodecagon.

f) Using this area and the area from the inscribed dodecagon found in the Looking Ahead portion, write an inequality that approximates pi.

g) Write an algorithm that can be used to find the area of an n-sided polygon that is circumscribed about a unit circle.

2. Use the "Lists and Spreadsheets" page on your graphing calculator to input the formulas from Problem 1 and find an approximation for pi for the n-sided polygons that Archimedes used. Write inequalities to approximate pi using the inscribed and circumscribed areas of the polygons.

n	**Inscribed Area**	**Circumscribed Area**
24		
48		
96		

Archimedes did all of these calculations by hand!

Section 5.14 Module Review

For Problem 1, find the limit and explain your reasoning.

1. $\lim_{x\to\infty} 100x^2 - x^3 + 10{,}000$

For Problem 2-4, find the limit.

2. $\lim_{x\to\infty} g(x)$ when $g(x) = 3x^3$

3. $\lim_{x\to\infty} h(x)$ when $h(x) = 5\sqrt{x}$

4. $\lim_{x\to\infty} j(x)$ when $j(x) = 10x - 4$

5. Demonstrate why $\lim_{x\to\infty} \frac{2x^4+1}{6x^2-x} = \infty$.

For Problem 6-10, solve the problem using $f(x) = \frac{x^2-3x+2}{(x+2)^2}$.

6. $\lim_{x\to\infty} f(x)$

7. $\lim_{x\to-2^-} f(x)$

8. $\lim_{x\to-2^+} f(x)$

9. $f(2)$

10. $f(-2)$

For Problem 11-13, let $f(x) = \frac{x^2+3x+2}{x-4}$ and $g(x) = \frac{x^2+3x+2}{x+1}$.

11. Which graph has an asymptote and what is it?

12. Which graph has a hole? Explain why this is a removable discontinuity.

13. Find the value of each function when $x = 1$.

For Problem 14 and 15, use the information given to solve the problem.

14. Evaluate the following limit:

$$\lim_{n \to \infty} (1 + \frac{0.6}{n})^n$$

15. Use $\mathrm{A} = \mathrm{P}e^{rt}$ to solve the following problems:

a) If \$300 is invested at 4.2% annual interest compounded continuously, how much will the investment be worth in five years?

b) Complete the steps to calculate when the investment will double.

$$\_\_\_\_\_\_\_\_ = 300 \cdot e^{0.042t}$$

$$2 = e^{0.042t}$$

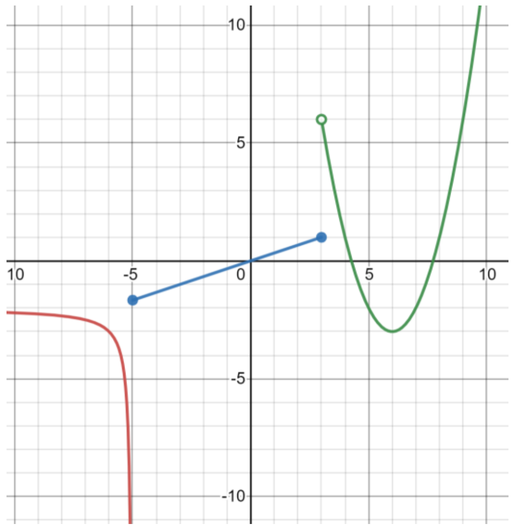
$$\ln 2 = \_\_\_\_\_\_\_\_ e^{0.042t}$$

$$\ln \_\_\_\_\_\_\_\_ = 0.042t$$

$$\frac{\ln 2}{0.042} = \_\_\_\_\_\_\_\_$$

$$\_\_\_\_\_\_\_\_ \approx t$$

For Problem 16-18, use the piecewise function $f(x)$ to solve the problem.



$$f(x) = \begin{cases} \frac{1}{x+5} - 2 \; \{x < -5\} \\ \frac{1}{3}x \; \{-5 \leq x \leq 3\} \\ (x-6)^2 - 3 \; \{x > 3\} \end{cases}$$

16. $f(-5)$

17. $\lim_{x \to 3} f(x)$

18. $\lim_{x \to \infty} f(x)$

For Problem 19 and 20, use any method to evaluate the limit.

19. $\lim_{x \to 1} \frac{x^2-4}{x+1}$

20. a) $\lim_{x \to 4^-} \frac{3}{x-4} + 2$

b) $\lim_{x \to 4^+} \frac{3}{x-4} + 2$

Section 5.15 Module Test

For Problem 1, find the limit and explain you reasoning.

1. $\lim\limits_{x\to\infty} 3.6^x - x^{1{,}000}$

For Problem 2-4, find the limit.

2. $\lim\limits_{x\to\infty} k(x)$ when $k(x) = (0.3)^x$

3. $\lim\limits_{x\to\infty} m(x)$ when $m(x) = \frac{5x^2+6x}{x^2-10}$

4. $\lim\limits_{x\to\infty} n(x)$ when $n(x) = \frac{100}{x}$

5. Demonstrate why $\lim\limits_{x\to\infty} \frac{5x^4+3x}{10x^2-3} = \infty$.

For Problem 6-10, solve the problem using $f(x) = \frac{x-5}{(x-3)^3}$.

6. $\lim_{x \to \infty} f(x)$

7. $\lim_{x \to -\infty} f(x)$

8. $\lim_{x \to 3^+} f(x)$

9. $\lim_{x \to 3^-} f(x)$

10. $f(10)$

For Problem 11-13, let $f(x) = \frac{x^2-5x+4}{x-4}$ and $g(x) = \frac{x^3+x}{x-4}$.

11. Which graph has an asymptote and what is it?

12. Which graph has a hole? Explain why this is a removable discontinuity.

13. Find the value of each function when $x = -2$.

For Problem 14 and 15, use the information given to solve the problem.

14. Evaluate the following limit:

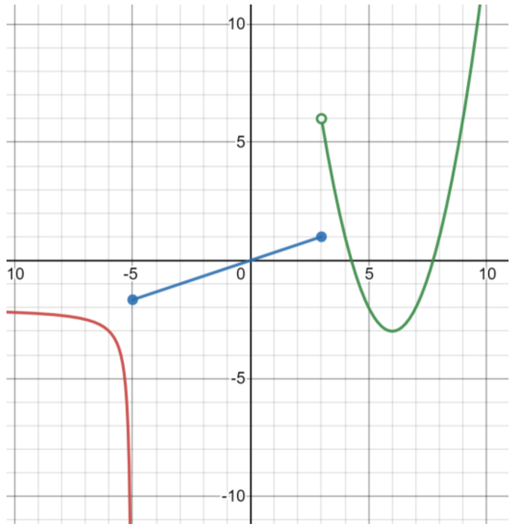
$$\lim_{n\to\infty}(1+\frac{9}{n})^{n}$$

15. Use Pe^{rt} to solve the following problems:

a) If $100 is invested at 1.75% interest compounded continuously, how much will the investment be worth in ten years?

b) When will the investment reach ten times the initial amount?

For Problem 16-19, use the piecewise function $f(x)$ to solve the problem.



$$f(x) = \begin{cases} \frac{1}{x+5} - 2 \; \{x < -5\} \\ \frac{1}{3}x \; \{-5 \le x \le 3\} \\ (x-6)^2 - 3 \; \{x > 3\} \end{cases}$$

16. $\lim_{x \to -5} f(x)$

17. $f(6)$

18. $\lim_{x \to 3^-} f(x)$

19. $\lim_{x \to 3^+} f(x)$

For Problem 20, use any method to evaluate the limit.

20. $\lim_{x \to \infty} \frac{1}{x-3} + 4$