Pre-Calculus and Calculus Module 8 Integrals and Integration

Section 8.1 Sequence and Series

Looking Back 8.1

Arithmetic sequences occur as terms proceed when a constant is added:

$4, 8, 12, 16 \ldots$

Arithmetic sequences are comparable to linear functions.

Geometric sequences occur as terms proceed when a constant is multiplied:

$4, 8, 16, 32 \ldots$

Geometric sequences are comparable to exponential functions.

Both of these are examples of discrete data. They both can model functions for compound interest, which do not rise continuously for a certain amount of money, but rather proceed in leaps.

Example 1: Answer the questions for the following sequence:

$7, 10, 13, 16 \ldots$

a) What are the next three terms in the sequence?

b) What is the value of the ninth term in the sequence?

c) Using a calculator, enter the term number on the x-axis and the value of the term on the y-axis. What type of equation models this?

The constant that is added to each term to get the next term in an arithmetic sequence is called a common difference. The recursive formula for the arithmetic sequence in Example 1 is $a_n = a_{n-1} + 3$ in which a_n is the term value (amount), n is the term number, and 3 is the common difference.

$\boldsymbol{n}$	1	2	3	4
$\boldsymbol{a_n}$	7	10	13	16

To use this formula, we must know the term before it. The explicit formula gives you any term value directly. The explicit formula is shown as follows:

$a_n = 7 + (n-1)3$ $\quad a_n = 7 + 3n - 3$ $\quad a_n = 4 + 3n$

Looking Ahead 8.1

Example 2: Audria invests in an account to save money for college. She deposits $\$600.00$ into the account that compounds annually with an interest rate of 8%. Use this information to solve the problems that follow.

a) Find the first four terms of the geometric sequence.

b) Let a_n be a function of n and graph a scatterplot of the data below.

c) What is the explicit formula for the amount of money (a_n) as a function of years (n)?

d) When will Audria have the $\$800$ needed for her first-year textbooks?

This is a geometric sequence. The constant multiplied by each term to get the next term is called the common ratio.

The formula for a geometric sequence in which a_1 is the first term and r is the common ratio is shown as follows:

$$a_n = a_1 \cdot r^{n-1}$$

Example 3: Use the graphing calculator to generate sequences with the recursive and explicit formula.

Section 8.2 Partial Sums and Series

Looking Back 8.2

Let us imagine a community garden needs to be planted; each volunteer can plant 4 plants a day and each day a new helper comes along to assist you. The first day it is just you, the second day it is you and one helper, and the third day it is you and two helpers, etc. Given you have 40 plants, how many days will it take to plant all of them?

Day 1 Total Plants: 4
Day 2 Total Plants: $4 + 8 = 12$
Day 3 Total Plants: $4 + 8 + 12 = 24$
Day 4 Total Plants: $4 + 8 + 12 + 16 = 40$

Given any sequence, we may proceed to add the second number to the first, the third number to the sum of the previous two, and so on. The sequence generates a series: $4 + 8 + 12 + 16$ ... The number of plants after each day is called a partial sum of the series: $4, 12, 24, 40$ ... The sum total you reach after n terms is called the nth partial sum of the series, usually written: "S_n." Because a partial sum S_n may usually be generated for each term in the series, the values of S_n will form their own sequence that may be analyzed.

Given a sequence a_n: $a_1,\ a_2,\ a_3,\ a_4$ ... which is a list of numbers that have some order, we may generate a series: $a_1 +\ a_2 +\ a_3 +\ a_4$ ... which is a sum of the sequence. Therefore, a sequence is a list of ordered pairs, while a series is a sum of numbers usually taken from some sequence.

With this new knowledge of finding partial sums, let us return to our plant problem.

Let us try to solve it another way. Add the first and last term of the sequence, then the second term to the second to last term, etc., and multiply the number you get by the number of pairs of terms.

$$S_4 = (4 + 16) + (8 + 12)$$
$$S_4 = 20 + 20$$
$$S_4 = 2(20)$$
$$S_4 = 40$$

The shortcut is to add the first and last term of the sequence, multiply by the total number of pairs of terms, which is the total number of terms divided by 2:

$$S_4 = \frac{4}{2}(4 + 16)$$
$$S_4 = 20 + 20$$
$$S_4 = 2(20)$$
$$S_4 = 40$$

The formula for the partial sum of an arithmetic sequence is $S_n = \frac{n}{2}(a_1 + a_n)$ in which n is the number of terms, a_1 is the first term, and a_n is the last term in the series. This can be written as follows:

$$S_n = \left(\frac{a_1 + a_n}{2}\right) \cdot n$$

If we put the partial sums in the graphing calculator for the first eight terms of the series, we get the perfect fit quadratic regression.

$y = 2x^2 + 2x$ and $f(4) = 2(4)^2 + 2(4) = 2(16) + 8 = 32 + 8 = 40$

Example 1: Find the sum of the first eight terms of the series:

$$3 + 10 + 17 + 24 + 31 + 38 + 45 + 52$$

Looking Ahead 8.2

The sequence below has a pattern as well:

n	1	2	3	4	…
a_n	3	8	15	24	…

For the n^{th} term, $a_n = (n+2)n$. A partial sum can be written using sigma notation:

$$S_4 = \sum_{n=1}^{4} (n+2)n$$

This notation represents the sum of $(n+2)n$ for the values of n from 1 to 4. Substitute $1, 2, 3,$ and 4 into n in the formula $(n+2)n$, then perform the operations and find the sum of the results. The number n is called the "term index" using sigma notation. The solution is shown as follows:

$$S_4 = \sum_{n=1}^{4} (n+2)n = 3 + 8 + 15 + 24 = 50$$

Example 2: Evaluate the $5th$ partial sum of S_5 for the series generated from the sequence $a_k = 2k - 3$:

$$\sum_{j=1}^{5} 2k - 3$$

Just as there is a shortcut for finding the sum of an arithmetic series, there is a shortcut for finding the sum of a geometric series. The 8th partial sum of a geometric series is shown below:

$$S_8 = 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$$

The first term, a_1, is 2, and the common ratio is $r = 2$. Let us multiply both sides of the equation by -2 and add it to S_8, our original series.

Example 3: Find the partial sum for the first eight terms of $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$ using the formula $S_n = a_1 \cdot \frac{1-r^n}{1-r}$ from Example 2.

A snail is traveling along a tree trunk to the ground that is 100 cm. away. Each pull forward he tries only moves him one-half the distance of the previous distance he moved. The lengths of his pulls are given below:

$$50 + 25 + 12.5 + 6.25 + 3.125$$

The snail will never complete the entire 100-centimeter distance, but partial sums will move him very close to it. The limiting value of the function is 100. We say this function converges to a limit of 100.

$$S_5 =$$

$$S_{10} =$$

$$S_{15} =$$

$$S_{20} =$$

Why does this happen? As n gets larger, $(\frac{1}{2})^n$ gets closer to 0; we can write this as a limit:

$$\lim_{n \to \infty} S_n =$$

If the common ratio is $|r| < 1$, the series will converge. If the common ratio is $|r| \geq 1$ and the first term a_1 is not equal to 0, then the partial sums do not approach a limit, but rather approach infinity, and the series diverges.

A series converges if it has a finite limit. A series diverges if it has no finite limit.

A convergent geometric series $|r| < 1$ converges to the value $S = a_1 \cdot \frac{1}{1-r}$ in which a_1 is the first term of the sequence and r is the common ratio.

In Module 2 of *Geometry and Trigonometry* (Data and Statistics), we learned that in the binomial expansion there are $(n+1)$ terms in the expanded expression for nth degree binomial $(a+b)^n$. Each term has the form Ca^sb^t in which $s+t=n$ for each term in which s and t are representative variables and C is the coefficient. The powers of a (the first term of the binomial) start at n and decrease by 1 for each term. The powers of b (the second term of the binomial) start at 0 and increase by 1 for each term. The coefficients, C, correlate to selections to form the nth row of Pascal's Triangle as we have previously seen.

1	$(a+b)^0$
1 1	$(a+b)^1$
1 2 1	$(a+b)^2$
1 3 3 1	$(a+b)^3$
1 4 6 4 1	$(a+b)^4$
1 5 10 10 5 1	$(a+b)^5$

As we learned previously, the coefficients of each term are a combination of n objects taken r at a time (r is the exponent of b). You may remember the formula ${}_n\mathrm{C}_r = \frac{n!}{(n-r)!r!}$ (so ${}_4\mathrm{C}_2 = \frac{4!}{(4-2)!2!} = \frac{4\cdot3\cdot2\cdot1}{(2\cdot1)(2\cdot1)} = \frac{12}{2} = 6$) from <u>Section 2.6</u> of *Geometry and Trigonometry*. Furthermore, the term containing b^2 is $6a^2b^2$. Because n is 4, the exponents have a sum of 4; this means a is squared as well. These ${}_n\mathrm{C}_r$ coefficients are called binomial coefficients. The expressions ${}_n\mathrm{C}_r$ is often written: "$\binom{n}{r}$" for short. The Binomial Theorem or Binomial Formula (also called Series of Theorem) captures all of these details.

The Binomial Theorem:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots \binom{n}{n-2}a^2b^{n-2} + \binom{n}{n-1}ab^{n-1} + b^n$$

This is for any numbers a and b in which $a \neq 0$ and $b \neq 0$ and n is any positive integer. Using Sigma Notation for the Binomial Theorem makes it the following equation:

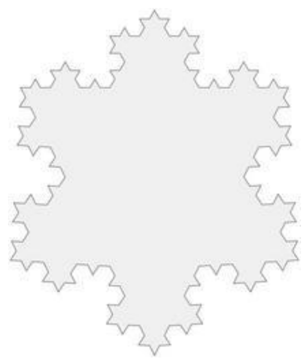
$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$$

Example 4:	Expand $(2n+m)^5$ using the Binomial Theorem.

Example 5:	Use sigma notation and summation to prove that 0.999 is equal to 1.

In the Practice Problems section today, you will also explore the Koch Curve, which is actually a fractal made out of line segments. The term "fractal" was used in 1975 by Benoit Mandelbrot to describe irregular recursions that occur in nature. These can be seen regularly in God's designs including ferns, trees, and coastlines of continents.

In 1904, Helge Von Koch, a Swedish mathematician, built his intrinsically pleasing Koch Curve, which has no tangent, by starting with an equilateral triangle. Each side is then divided into three equal parts and the middle part is replaced by building another equilateral triangle with the deleted portion of the segment as its base, and the previous base erased. This iteration led to the Koch Snowflake, which is also called Koch Island.



Fractals are self-similar. The parts of the object are copies of the objects. The pieces are the same shape as the whole, but different sizes.

The cauliflower is a vegetable in which fractal properties can be observed. Each branch resembles the original head. Broccoli is another vegetable in which branches of self-similarity can be observed.

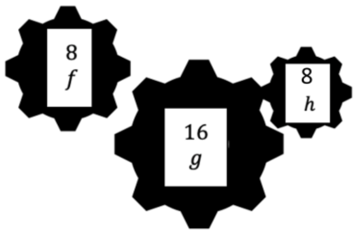
Mandelbrot did not make these vegetables or the coastland, but he did discover the patterns that God created in them. Mathematics is used to define and describe natural phenomena that occur on earth. Like the God who created it, mathematics is omnipresent: everywhere around us. Mathematics can use language, and invent notation and symbols to define and describe what is discovered in the natural world around us… However, "it is above the power of humans to alter, direct, or influence it," Mario Livio writes of mathematics in his work *Is God a Mathematician?*

Section 8.3 The Derivative and the Indefinite Integral

Looking Back 8.3

In the previous module, we learned how to apply the Chain Rule as a method of differentiation. We differentiate the function on the outside and then differentiate the function on the inside:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$



The Chain Rule works much like the cogs on a gear. Notice that when f makes one full rotation of 8 cogs, g has only made one-half of a rotation (8 out of 16 cogs). When h makes one full rotation (8 cogs), g has only made one-half of a rotation; one thing has an effect on the other. These gear ratios represent a composition of functions.

In mathematics, many things are connected. In two-variable problems, the independent variables (called "explanatory variables" in Statistics) have an effect on the dependent variables (called "response variables" in Statistics).

In Module 6, we investigated velocity and position graphs, and discovered the first derivative of the distance function is the velocity function. We also discovered the area under the velocity curve is the total distance an object traveled.

In the previous module, Module 7, we spent time finding the slope of tangent lines and deriving a shortcut called the derivative.

In this module, we will use prior knowledge to find the area under the curve and then use a process called Riemann Sums to eventually derive a shortcut called the integral.

Looking Ahead 8.3

Let us do a little review of our velocity and distance functions and how they are related.

Example 1: Yolanda has a long drive to get to a conference. She puts her car on cruise control at 4:00 pm. The cruise control is set at 60 miles per hour. At 6:00 pm, she turns off the cruise control and exits the highway to go to the college where the conference is being held. Use this information to solve the problems below.

a) Draw a graph of Yolanda's velocity.

b) What is the equation for the velocity function of t hours in which t is the time elapsed since 4:00 pm?

c) What is Yolanda's acceleration (increase or decrease in velocity)?

d) How can you find Yolanda's total distance traveled from 4:00 pm to 6:00 pm from the velocity graph?

Example 2: Although the velocity is constant for Yolanda's trip between 4:00 pm and 6:00 pm, her distance traveled steadily increases. Use this information to answer the questions below.

a) What is Yolanda's rate of speed during the two-hour interval?

b) What is the distance function for Yolanda's cruise control travel for time t? Draw the graph over the two-hour interval of cruise control. Write the distance traveled by Yolanda since 4:00 pm as a function of t: $d(t)$.

c) What is the slope of the line for $d(t)$ at any point on Yolanda's distance function?

d) How is Yolanda's distance function related to the velocity function in Example 1?

$$d(t) = \int v(t)\,dt \text{ and } v(t) = d'(t)$$

The notation on the right is familiar to us from the discoveries we made in the last module. We know how to find the derivative of $d(t)$ using the Power Rule. Because $d(t)$ is equal to $60t$, $d'(t)$ is equal to 60 and $v(t)$ is equal to 60.

The notation on the left is new. The symbol "$\int$" is the integral symbol and represents the area under the velocity curve. It is not hard to find the area under the curve when it is a shape that we know such as a rectangle or a triangle; however, a curve can be difficult.

In this module, we will learn how to find the area under a curve that is not a straight horizontal or diagonal line. We will develop methods to find the integral (integrate) just as we previously found methods to find the derivative (differentiate).

To go backward from the derivative of a function, $d'(t)$, to the function itself, $d(t)$, is called antiderivative. The antiderivative and the indefinite integral are closely related and will be explored further in Section 8.11.

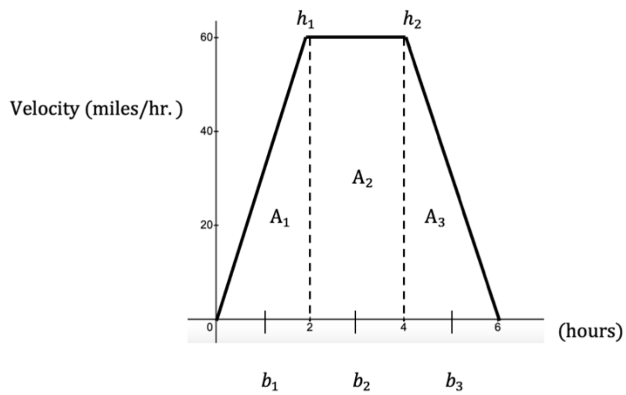
In Section 8.4 to 8.6, we will investigate area under a curve using Riemann sums. This will lead to Section 8.9 to 8.11 in which we will investigate the Fundamental Theorem of Calculus and further study the definite and indefinite integral, and antiderivatives.

Section 8.4 Riemann Sums

Looking Back 8.4

Yolanda travels at an increasing constant rate for two hours until she reaches 60 miles per hour and then sets the cruise control in her car for the next two hours. After the two hours, she travels at a decreasingly constant rate for another two hours until she gets to her conference. How far has Yolanda traveled?

To find the area under the curve, we can draw the graph and add up the area of the pieces under the curve, which are made up of triangles and rectangles.



$$\text{Total Area} = A_1 + A_2 + A_3$$

$$\text{Total Area} = \frac{1}{2}(b_1)(h_1) + b_2(h_1 \text{ or } h_2) + \frac{1}{2}(b_3)(h_2)$$

$$\text{Total Area} = \frac{1}{2}(2)(60) + 2(60) + \frac{1}{2}(2)(60)$$

$$= \frac{1}{2}(120) + 120 + \frac{1}{2}(120)$$

$$= 60 + 120 + 60$$

$$= 240 \text{ miles}$$

Adding up the area of the pieces of the known shapes results in the total area under the curve. In this case, it is the total distance Yolanda traveled. When the shapes are unfamiliar, we can break up the area under the curve into smaller and smaller pieces until it approximates the area of a known shape. Breaking up the area under the curve into pieces and adding the areas of all the pieces together is known as a Riemann sum.

Riemann's sum gets its name from the German mathematician Bernhard Riemann. Growing up, Riemann was schooled by his father, so he enrolled at the University of Göttingen to study theology in honor of his father's wishes. However, Bernhard wanted to study mathematics. He asked his father if he could switch studies out of respect and permission was granted.

In 1849, Riemann worked on his doctorate thesis with the assistance of Carl Gauss, another German mathematician who made contributions to his field, and a man who recognized Riemann's genius. This genius is what made Riemann well known for his work on the generalized Riemann Hypothesis and Riemann's zeta functions. Attempts to prove the Riemann Hypothesis can be found in the movie *A Beautiful Mind,* Season 1 of *Prime Suspect,* the crime drama *NUMB3RS,* and the novel *Life After Genius* published in 2008 and written by M.A. Jacoby.

In Calculus, the Riemann sum is of the form $\sum f(x) \cdot \Delta x$. One common application of the Riemann sum is the approximation of the area under the curve of a function. Though many procedures may be used to find area under the curve, the Riemann sum is the strongest and may be applied to nearly any function.

Looking Ahead 8.4

Remember that the symbol Σ is the capital Greek letter sigma and is used to indicate sums. This sigma notation is a shorter way to write a long list of numbers as terms that are being added together.

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5 = 15$$

Remember also that this is read: “The sum of k from k equals one to k equals five.” As you can see, the sum is equal to 15. The k under the sigma symbol is called the index. The k next to the summation symbol is called the argument and k must always be an integer value; k cannot include fractions or decimals. Start with the initial index value under the sigma and end with the final index value above the sigma.

Example 1: Find the value of the sum below, which is written:

$$\sum_{k=3}^{5} k^2$$

Example 2: Write the sum below in expanded form and calculate the value:

$$\sum_{n=1}^{3} \frac{n}{n+1}$$

Example 3: Answer the following questions for the given sequence.

$1.3, 1.6, 1.9, 2.2, 2.5, 2.8, 3.1$

a) What is the initial value of the expression?

b) What is being added each time?

c) If k starts at -1, what does k end at?

d) Explain why the kth term in the sequence a_k may be found using the expression $a_k = 0.3k + 1$?

An infinite series goes on forever, such as $\sum_{k=1}^{\infty} 3k$ in which a_k is equal to $3k$. The first four terms of the sequence a_k are $3, 6, 9, 12$. We can get partial sums, denoted S_4, for the first four terms, which would be $3 + 6 + 9 + 12 = 30$. The infinite series for partial sums is S_∞. We can write $\sum_{k=1}^{\infty} 3k = S_\infty$ for the sum of infinite terms; however, a better way to write this is $\sum_{n=1}^{\infty} 3k = \lim_{k\to\infty} S_k$.

If the sequence approaches a limit, $\lim_{k\to\infty} a_k = \text{L}$, then the sequence a_k converges, but if the limit does not exist, then the sequence a_k diverges and includes cases in which the limit is not finite. If the limit goes to $\pm\infty$ (increases or decreases without bound) it diverges. The same is true of a series. If $\sum_{k=1}^{\infty} a_k$ is equal to S, the series a_k has a sum the number S, then it converges; however, if a series $\sum_{k=1}^{\infty} a_k$ is equal to $\pm\infty$, then it diverges. We cannot get a specific number or a finite value for the limit.

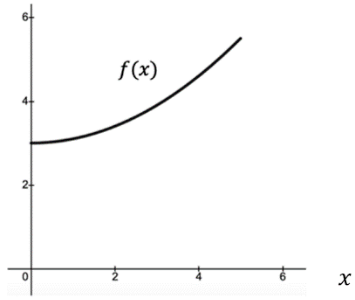
Example 4: Does $\sum_{k=1}^{\infty} 3k$ converge or diverge?

A formula for partial sums of an arithmetic series in terms of k is $S_k = \left(\frac{a_1 + a_k}{2}\right) \cdot k$. So, for this example, S_k is equal to $\left(\frac{3+3k}{2}\right) k$. Now, $\lim_{k\to\infty} S_k$ is equal to $\lim_{k\to\infty} \left(\frac{3+3k}{2}\right) k$, which is two infinitely large numbers being multiplied. The series definitely diverges.

Section 8.5 Left and Right Endpoint Rectangles for Approximating Area

Looking Back 8.5

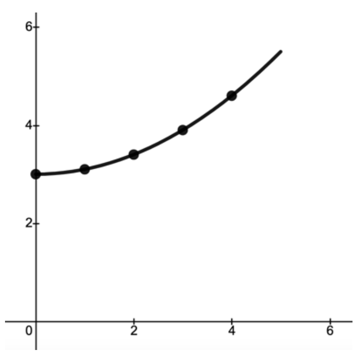
How can we find the area under the function $f(x) = \frac{1}{10}x^2 + 3$ from 0 to 5 seconds to approximate total distance? This area is not in the shape of a rectangle or triangle, but a quadrilateral with a curved top.



One way to approximate total distance is to break up the area under the curve into smaller rectangular shapes and add up all the areas of these rectangles.

The region is bounded below by the x-axis. It is bounded above by the function $f(x) = \frac{1}{10}x^2 + 3$. It is bounded on the left by $x = 0$ and bounded on the right by $x = 5$; this could be written as an inequality interval: $0 \le x \le 5$. If we let A represent area, then we can say we are trying to find the area A of f when $0 \le x \le 5$. This can be written "A($f, 0 \le x \le 5$)." On the other hand, because $f(x)$ is equal to $\frac{1}{10}x^2 + 3$, it could be written "A($\frac{1}{10}x^2 + 3, 0 \le x \le 5$)."

If we break the area under the curve into rectangles whose widths are 1 unit each ($\Delta x = 1$), then the bases of the rectangles would be between 0 and 1, 1 and 2, 2 and 3, 3 and 4, and 4 and 5. The height of each rectangle is the length of y, $f(x)$. If we let $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $x_4 = 4$, then what should the height of each triangle be? One possible choice, called the left-endpoint method, is to pin the top left corner of each rectangle to the curve of the function. Then the corresponding heights would be $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$, and $f(x_4)$.



We can draw dashed lines to represent rectangles at each of the given points on the graph. To find the area of a rectangle, only the length of the base and length of one side is needed because opposite sides are equal. We do not need to use $f(x_5)$ as the left edge of the rectangle because it is in the interval $n - 1$ for the left-endpoint. The rectangles are under the curve so the estimated area will be an underestimate.

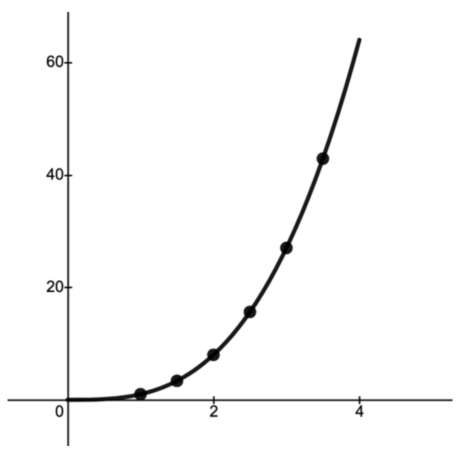
Because the base of each rectangle is the change in x (Δx) and the height is $f(x)$, a general formula for area of one rectangle is A $= \Delta x \cdot f(x)$.

If x_k represents the left endpoint of each interval and there are n intervals, we add up all the areas from $k = 0$ to $k = n - 1$; we are finding $\sum_{k=0}^{n-1} f(x)\Delta x$ to get the total area.

Let us find A($f, 0 \le x \le 5$) or A($\frac{1}{10}x^2 + 3, 0 \le x \le 5$) to see if the general formula really works for finding the area under the curve $\frac{1}{10}x^2 + 3$ when x is in the interval $[0, 5]$.

Example 1: Find the area under the function $\frac{1}{10}x^2 + 3$ over the interval $0 \leq x \leq 5$ using left-endpoint rectangles.

Example 2: The graph of $f(x) = x^3$ for $0 \leq x \leq 4$ is drawn, but this time the interval between $x = 1$ and $x = 4$ is divided into 6 equal segments. Use the graph to complete a)-h).



c) What is the width of each rectangle?

b) Does the width represent the base or the height of the rectangle?

c) Fill in the table below.

Term or Interval Number (k)	0	1	2	
x-value of Left Endpoint of Interval (x_k)	1	1.5	2	2.5

d) List the coordinates of each darkened circles for $(x_k, f(x_k))$ from $k = 0$ to $k = 3$.

e) In terms of rectangles, is $y = f(x)$, the base or the height?

f) What is the area of A_1?

g) If using the left-endpoint method to determine the area for the entire interval from $x = 1$ to 4, will k end up going from 0 to 3, 0 to 4, or 0 to 5?

h) Write out the sum of the areas to get the total area under the curve of the function from $k = 0$ to $k = 5$.

Writing the function in summation notation is a little trickier in this instance. We must find x in terms of k. The start value for x is 1 when k is 0 so x_0 equals 1. From 1, we add 0.5 to get to 1.5. From 1, we add 0.5 twice to 2.0. From 1, we add 0.5 three times to get to 2.5. This would be 0.5(1) or $0.5(2)$ then $0.5(3)$. We are multiplying 0.5 by the value of k, the index at that point, to get the entire sum. Mathematically, this looks like $1 + 0.5k$.

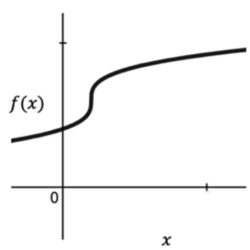
To find the y-values at each of those x-values, we must cube the expression: $(1 + 0.5k)^3$. Though y (the height of the rectangle) is changing with each value for x, the base of the rectangle is always 0.5. Therefore, the area is $0.5(1 + 0.5k)^3$, or to be more mathematically correct, $0.5(0.5k + 1)^3$. Using summation notation, the area can be found by finding the sum of the function:

$$\sum_{k=0}^{5} 0.5(0.5k+1)^3$$

This can also be written $0.5 \sum_{k=0}^{5} (0.5k+1)^3$ because the sum of the entire area can be multiplied by 0.5 rather than the sums of each of the areas.

In more general terms, $\mathrm{A}(f, a \le x \le b)$ is the area under the curve $f(x)$ between a and b and including a and b.

If each rectangle has a width of Δx (the change in x each time between intervals), then using the left-endpoint method…



$$\mathrm{A}(f, a \le x \le b) = \sum_{k=0}^{n-1} \Delta x \cdot f(a + \Delta x \cdot k)$$

This can also be written…

$$\mathrm{A}(f, a \le x \le b) = \Delta x \sum_{k=0}^{n-1} f(a + \Delta x \cdot k)$$

… in which a is the start of the entire interval, b is the end of the entire interval, k is the index number, and n is the number of intervals.

We have already learned that to find Δx, we can find the length of the entire interval and divide by the number of smaller intervals we are looking for.

$$\Delta x = \frac{b-a}{n}$$

If we substitute this into our formula from above, we get the following formula:

$$\mathrm{A}(f, a \le x \le b) = \sum_{k=0}^{n-1} \frac{b-a}{n} f(a + \frac{b-a}{n} \cdot k)$$

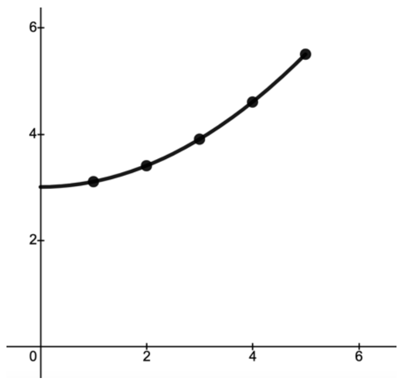
This can also be written as follows:

$$\mathrm{A}(f, a \le x \le b) = \frac{b-a}{n} \sum_{k=0}^{n-1} f(a + \frac{b-a}{n} \cdot k)$$

Now we have everything in terms of a, b, k, and n rather than x, which is helpful because the values of a, b, k, and n are known to us at the beginning of the problem.

The symbol x_k represented each interval for n intervals. We summed up all the areas of the rectangles from $k = 0$ to $k = n - 1$ using the left endpoint rectangle method. Let us compare this to the right-endpoint rectangle method, which you can likely guess involves pinning the right endpoint of each rectangle to the function.

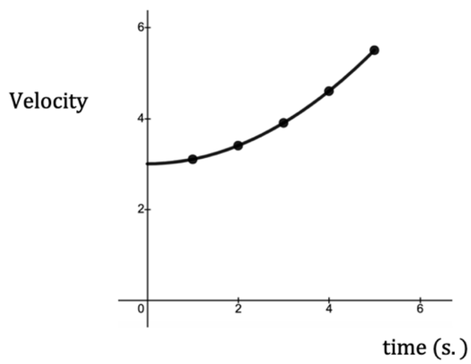
Looking Ahead 8.5



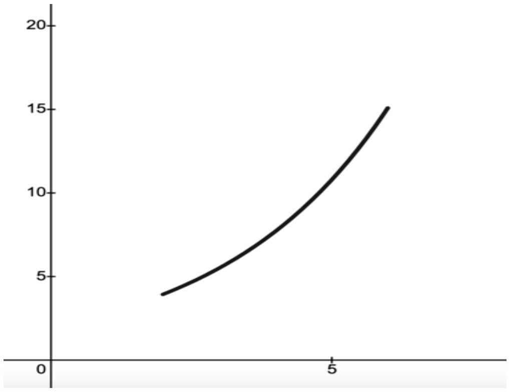
There are still five intervals on the graph: x_0, x_1, x_3, x_4, and x_5. We are looking at the lengths of the right-most sides of the rectangles to find the area under the curve: $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$, and $f(x_5)$. We then sum up the total area of the rectangles from $k = 1$ to $k = n$.

Most of the rectangles go above the curve so this will be an overestimate of area whereas the left-endpoint method produced rectangles that had areas mostly below the curve and therefore resulted in underestimates of area.

Example 4: Find the total distance traveled by an object from 0 to 5 seconds using the right-endpoint method with five rectangles if the velocity as a function of time is $\frac{1}{10}x^2 + 3$.



Example 5: Bacteria grow at a rate modeled by the function $g(x) = 2 \cdot 1.4^x$ per day for x days. Find the summation formula for the total number of bacteria cultured in a scientific laboratory between 2 and 6 days using 8 sub-intervals and the right-endpoint method.

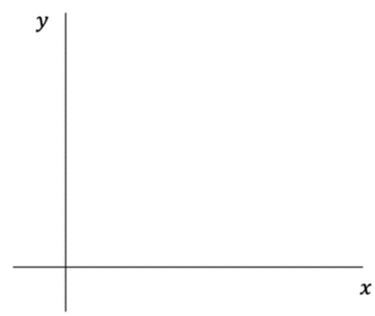


For left-endpoint rectangles, the approximate area under the curve $y = f(x)$ is found by dividing the interval into n sub-intervals and using the left-most endpoint of each interval.

$\sum_{k=0}^{n-1} f(x_k)\Delta x$ (in function and interval notation)

$\Delta x \sum_{k=0}^{n-1} f(a + \Delta x k)$ (in terms of a, Δx, and k)

$\frac{b-a}{n} \sum_{k=0}^{n-1} f(a + (\frac{b-a}{n}) \cdot k)$ (in terms of a, b, n, and k)

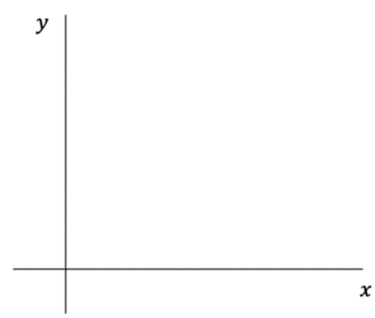


For right-endpoint rectangles, the approximate area under the curve $y = f(x)$ is found by dividing the interval into n sub-intervals and using the right-most endpoints of each interval.

$\sum_{k=1}^{n} f(x_k)\Delta x$ (in function and interval notation)

$\Delta x \sum_{k=1}^{n} f(a + \Delta x k)$ (in terms of a, Δx, and k)

$\frac{b-a}{n} \sum_{k=1}^{n} f(a + (\frac{b-a}{n}) \cdot k)$ (in terms of a, b, n, and k)

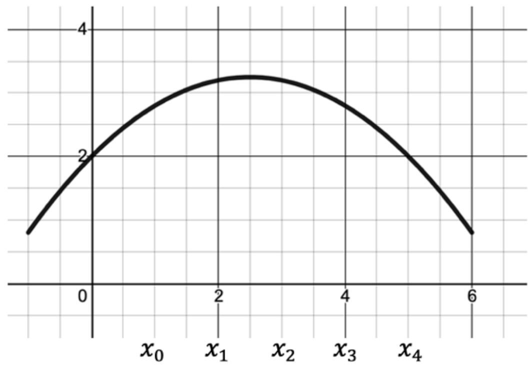


The function notation tells us what is being done but not how it is being done. The sigma notation uses specific assigned values so we can calculate Δx when a, b, and n are known. The beginning, end, and number of intervals are given information.

Section 8.6 Midpoint and Trapezoidal Methods

Looking Back 8.6

In this section, we will use midpoint rectangles to find the sum of the area under a curve. First, we must find the midpoint of the rectangles. We will start halfway between x_0 and x_1 to find the height of the first rectangle, and end halfway between x_3 and x_4 to find the height of the fourth rectangle. Then we will draw a vertical line from the midpoint to the curve. Next, we will draw a horizontal line across the rectangle height that extends to the interval on the left (x_0 or x_3) and the right (x_1 or x_4).



Previously, when we used left-endpoint rectangles to find the area under the curve, we got an underestimate. When we used right-endpoint rectangles to find the area under the curve, we got an overestimate. If we take the average of the two areas, we can get a good approximation for the area under the curve:

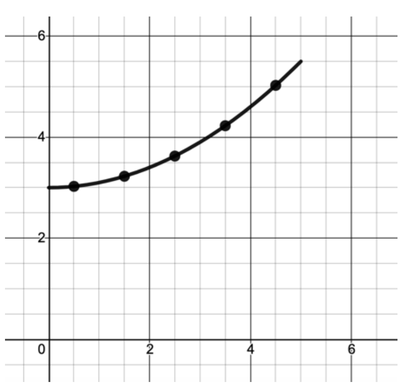
$$\frac{\text{Lower Bound} + \text{Upper Bound}}{2}$$

$$= \frac{18 + 20.5}{2}$$

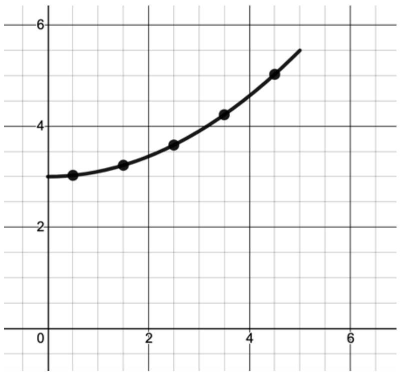
$$= 19.25$$

An area of 19.25 sq. units is a better approximation of displacement. Do you think using midpoint rectangles would give us this average area? Let us try it and see.

Example 1:	Using midpoint rectangles, find $A(\frac{1}{10}x^2 + 3, 0 \leq x \leq 5)$ for five intervals.



Example 2: Write the area under the curve $\frac{1}{10}x^2 + 3$ using sigma notation and Riemann sums with midpoint rectangles.



Now we have the same problem as in Example 1 and the sum is 19.125. Now we also have a rule for midpoint rectangles:
The approximate area under the curve $y = f(x)$ is found by dividing the interval into n sub-intervals and using the midpoints of each interval.

$$\sum_{k=0}^{n-1} f(x_k)\Delta x$$ (in function and interval notation)

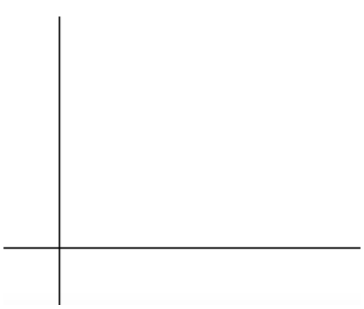
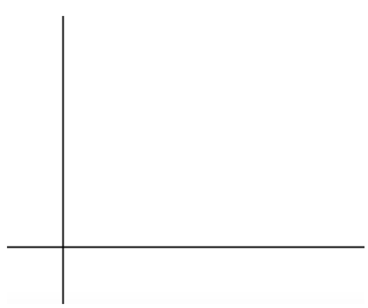
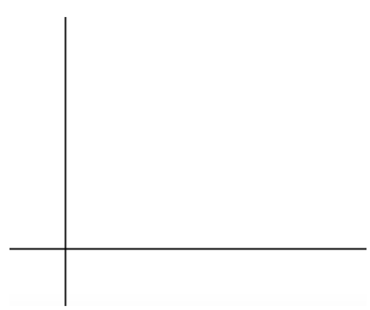
$$\Delta x \sum_{k=0}^{n-1} f\left(a + \Delta x\left(\frac{1}{2} + k\right)\right)$$ (in terms of a, Δx, and k)

$$\frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{b-a}{n}\left(\frac{1}{2} + k\right)\right)$$ (in terms of a, b, n, and k)

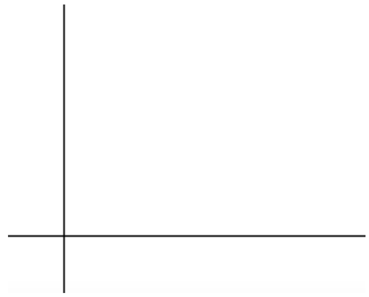
Now that we know how to find the area under the curve using midpoint rectangles, we will learn how to find the area under the curve using trapezoids.

You probably remember learning about Bernhard Riemann in Section 8.4. In this section, we will learn how to solve Riemann's sum by using trapezoidal sections.

The more sub-intervals, n, that an interval $[a, b]$ is divided into, the better the approximation will be because the smaller and smaller rectangles more closely fit the area under a curve.

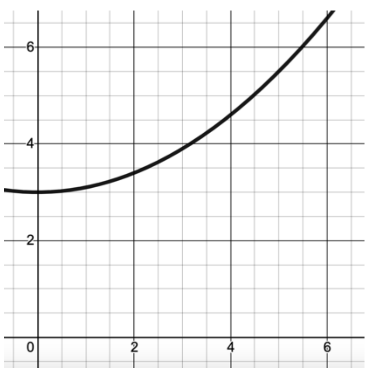




If we draw a shape between the bases and the left and right-endpoint of each sub-interval under the curve, we see that our shapes approximate the area of trapezoids rather than rectangles.



The area formula for a trapezoid is $A = \frac{1}{2}(b_1 + b_2) \cdot h$. This time, Δx becomes the height of the trapezoid. This height is perpendicular to each base, so the bases are the left and right-side length of each interval. For example, the bases for the first sub-interval are $b_1 = f(x_0)$ and $b_2 = f(x_1)$.

Example 3: Find the area under the curve $\frac{1}{10}x^2 + 3$ from 0 to 5 seconds using the trapezoids and five intervals.



It might be easier to find the function at each sub-interval first as they are repeated for adjacent trapezoids that share a common side.

The area under the velocity curve $\frac{1}{10}x^2 + 3$ was 18 when we used left-endpoint rectangles.

The area under the velocity curve $\frac{1}{10}x^2 + 3$ was 20.5 when we used right-endpoint rectangles.

The area under the velocity curve $\frac{1}{10}x^2 + 3$ was 19.25 when we used trapezoids.

Now, make a conjecture based on logic and observation about the methods for finding the area under a curve.

The area of the velocity function $\frac{1}{10}x^2 + 3$ was 19.125 when we used midpoint rectangles. We found the average of the left and right-endpoint rectangles, $\frac{18+20.5}{2} = 19.25$, and determined the midpoint rectangle area was not equal to that. However, the area was 19.25 when we used trapezoids.

Section 8.7 Negative Area Under the Curve

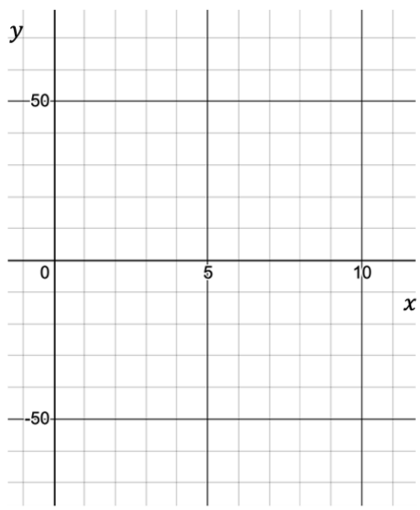
Looking Back 8.7

The central idea of differential calculus is the derivative. We previously explored tangent and velocity problems to introduce derivatives. Now, we are computing area and solving distance problems. As the number of rectangles increases, the exact area is the limit of the sums of the areas. This is used to introduce the definite integral and integral calculus. In this section, we will explore what negative area looks like, how it can be summed up, and what it means.

In the last module, we learned that speed is how fast an object is moving. The direction of movement is not included in speed. However, velocity has both magnitude and direction. It can be positive or negative. The absolute value of velocity $|v|$ is speed. Like absolute value in general, speed can never be negative.

Looking Ahead 8.7

Example 1: Let the velocity be positive when a car is traveling away from home. Let the velocity be negative when a car is traveling toward home. Find the total displacement of a car that travels 70 miles per hour for three hours and then 50 miles per hour for one hour to arrive at an appointment that lasts an hour. Then the car travels home by a different route, 60 miles per hour for two and a half hours and then 55 miles per hour for two hours.



The velocity graph for this car shows a constant rate for three hours out, then one hour out. It is positive because the car is traveling away from home. The constant rate for the next two hours and a half and another constant rate for the next two hours is negative because the car is returning home.

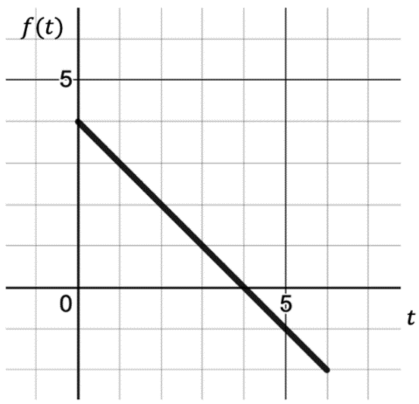
Adding up the area under the velocity curve will result in the total displacement of the car; A_1 and A_2 will represent positive displacement while A_3 and A_4 will represent negative displacement.

Example 2: A ball is rolling up a hill at a velocity of $f(t) = 4 - t$ centimeters per second for 4 seconds. Find the areas below:

a) $\mathrm{A}(f(t), 0 \leq t \leq 4)$

b) $\mathrm{A}(f(t), 4 \leq t \leq 6)$

What do these areas tell us?



Section 8.8 Using Technology to Calculate Riemann Sums

Looking Back 8.8

We have investigated four methods to find area under the curve: left-endpoint rectangles, right-endpoint rectangles, midpoint rectangles, and trapezoids. We have taken the function for the curve and converted it using sigma notation in order to use Riemann sums to find the area of the rectangles for all the sub-intervals over an entire interval.

As a function becomes more complex, it becomes more complex to make calculations by hand using sigma notation and Riemann sums.

However, the graphing calculator has a feature that allows this process to be completed more quickly. Of course, we have to know what to do in order to input the proper sigma notation into the calculator to find the Riemann sum.

Looking Ahead 8.8

Example 1: The first example of Riemann sums in this module was in Example 1 in Section 8.1:

$$\sum_{k=3}^{5} k^2 = 3^2 + 4^2 + 5^2 = 9 + 16 + 25 = 50$$

Use the calculator to check if the sum is actually 50.

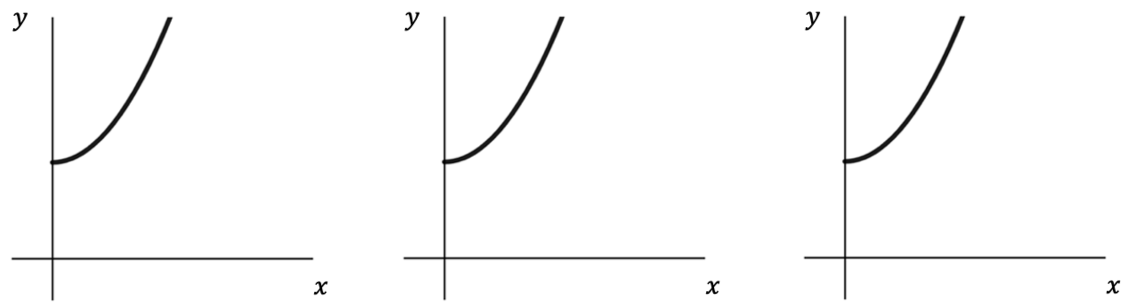
Example 2: In Example 4 of Section 8.3, left-endpoint rectangles were used to find the area under the curve $f(x) = x^3$ over the interval $[0, 3]$ using six sub-intervals. When we calculated this by hand in Section 8.3, we found the area to be approximately 48.9375. Check this using Riemann sums.

Example 3: In Example 3 of Section 8.4, right-endpoint rectangles were used to find the number of bacteria between 2 and 6 days over eight sub-intervals when the growth was modeled by the function $g(x) = 2 \cdot 1.4^x$ and x was the number of days. Using $\Delta x \sum_{k=1}^{n} f(a + \Delta x k)$, the sigma notation became $0.5 \sum_{k=1}^{8} 2 \cdot 1.4^{(2+0.5k)}$. This works! The total bacteria were approximately 35.9683. How does changing the number of sub-intervals to 20 change the calculated area? Find the proper sigma notation to represent this area and use the calculator to find the Riemann sums.

Section 8.9 Antiderivatives and the Definite Integral

Looking Back 8.9

No matter how many rectangles are drawn under a curve, the area can only be approximated; $\frac{b-a}{n}$ represents Δx. As the number of intervals (n) increases, the width of each rectangle (Δx) decreases.



One way to get an exact area is to take the limit of the sum of the areas as $n \to \infty$. As n gets bigger and bigger, Δx gets smaller and smaller. As the limit of the sum of the areas $n \to \infty$, then $\Delta x \to 0$.

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f\left(a + \frac{b \cdot a}{n} \cdot k\right)$$

This is a Riemann sum that represents the exact area under a curve.

As the area of each rectangle approaches zero, the overall area reaches a limit, but not zero because the sum of all the areas is being added. The limit predicts a sum as n approaches infinity, but n cannot ever be equal to infinity. Only God reaches infinity.

We have investigated the area under a curve using four estimation methods. To get an exact area, we find the limit as the number of rectangles approaches infinity. The limit of the Riemann sum is called the definite integral.

The area under the function f over the interval $[a, b]$ is shown as follows:

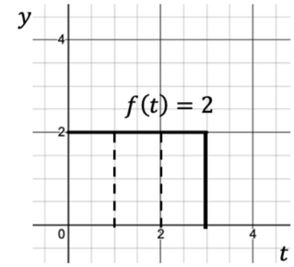
$$\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(a + \Delta x \cdot k) = \int_a^b f(x)dx$$

The integral sign is the symbol $\int$, the $f(x)$ next to the integral sign is called the integrand, and Δx becomes dx in the integral. The lower bound of the interval is a and the upper bound of the interval is b. This is the start and endpoint of the area. The dx becomes an infinitely thin width and $f(x)$ represents all the heights of the rectangles that have dx as a base.

Looking Ahead 8.9

Example 1: Find the areas of the integrals below. A capital letter A will represent the area under the curve.

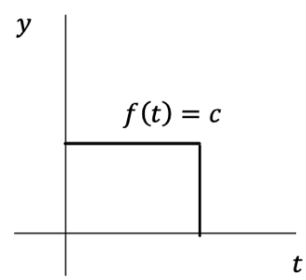
a) $A(0) = \int_{0}^{0} 2dt$



b) $A(1) = \int_{0}^{1} 2dt$

c) $A(3) = \int_{0}^{3} 2dt$

d) $A(x) = \int_{0}^{x} 2dt$

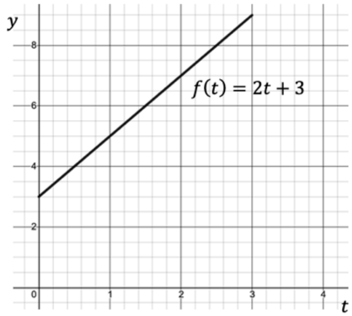


We can generalize a rule based on the patterns we have found.

$A(x) = \int_{0}^{x} c\,dt = c \cdot x$ (in which c is any constant)

Example 2: Find the areas of the integrals below when the function is not a linear constant function, but a linear function that is not a constant. In Example 1, m is equal to 0, but for these linear functions m is not equal to 0.

a) Find the area of the definite integral $\int_0^1 (2t+3)dt$.



b) Find the area of the definite integral $\int_0^4 (2t+3)dt$.

c) Find the area $\mathrm{A}(x) = \int_0^x (2t+3)dt$. This is the area under the curve when $0 \le t \le x$.

The limit of the Riemann sum is the definite integral. You can find the area under the curve in other ways, but this gives you a way to always find it.

The equation $\mathrm{A}(x) = x^2 + 3x$ is the exact area under the curve. The shape of this area is a trapezoid.

Look at the integrand function $f(t) = 2t + 3$, which also means $f(x)$ is equal to $2x + 3$. How does this function relate to the function $\mathrm{A}(x) = x^2 + 3x$? The generalized area function may be written $\mathrm{A}(x) = \int_0^x (mt+b)dt$.

The function $2t + 3$ is linear and is the equation of a line with a slope of 2 through the point $(0, 3)$. Remember, the slope of the tangent line at a point on the position curve is velocity. The area under the velocity curve is total distance. The velocity is the derivative of the distance function.

$$\mathrm{A}(x) = x^2 + 3x \qquad \mathrm{A}'(x) = 2x + 3$$

The derivative is the integrand. We can use reverse thinking to find the area function. This is called the antiderivative.

Example 3: Use the given information to evaluate $A_3(x) = \int_1^x (2t+3)dt$. (In doing this we will be able to see what happens to a function when the lower bound is not 0.)

$$A_1(x) = \int_0^1 (2t+3)dt = 4$$

This is called a definite integral. It has a numerical answer.

$$A_2(x) = \int_0^x (2t+3)dt = x^2 + 3x$$

This is called an indefinite integral. It has a variable answer.

If the interval of the indefinite integral is from a fixed constant to some variable x-value that is unknown, then we can generically say the following equation is true:

$$A_3(x) = A(x) - A(c)$$

If $A(x)$ is equal to $\int_a^b f(x)dx$, then $A(x)$ is equal to $f(b) - f(a)$.

This is how we evaluate the definite integral between two constants when the lower bound is not 0:

$$\int_2^4 (2t+3)dt = t^2 + 3t \Big|_2^4$$
$$= \left(4^2 + 3(4)\right) - (2^2 + 3(2))$$
$$= (16+12) - (4+6)$$
$$= 28 - 10$$
$$= 18$$

First, find the antiderivative. Substitute the upper bound of the interval in the antiderivative function and evaluate it. Then substitute the lower bound of the interval in the antiderivative function and subtract it from the value you calculated for the upper bound.

Finding the sum of the areas between two boundaries is the same as the difference between the area of the upper bound and the area of the lower bound.

Section 8.10 The Fundamental Theorem of Calculus

Looking Back 8.10

The indefinite integral of a function is its antiderivative. This fact is called The Fundamental Theorem of Calculus and it will let us solve integral problems without using Riemann sums. The derivative and the integral are inverse processes. If you take the derivative of an integral, you get the function you started out with. Any letter may be used for a function; for example, $f(t)$, $g(x)$, $\mathrm{A}(x)$, …, etc. The antiderivative of a function results in a new function whose derivative is $f(x)$. Let us call this $\mathrm{F}(x)$.

$$\frac{d}{dx}\mathrm{F}(x) = f(x)$$

And

$$\int f(x)dx = \mathrm{F}(x)$$

There are an infinite number of antiderivatives $\mathrm{F}(x)$ that differ only by a constant, so we write the functions as follows:

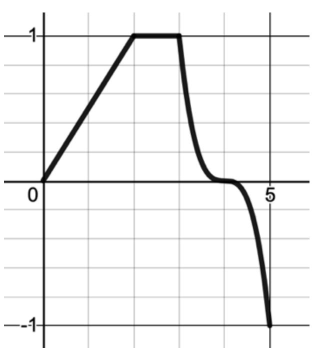
$$\frac{d}{dx}\mathrm{F}(x) + c = f(x)$$

And

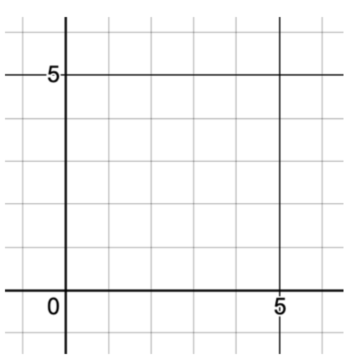
$$\int f(x)dx = \mathrm{F}(x) + c$$

We will first review two theorems that will help us understand the Fundamental Theorem of Calculus.

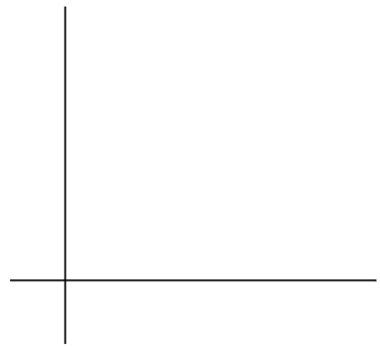
Example 1: Let f be the function shown in the graph. Let $\mathrm{A}(x)$ be the area under the curve. The graph shows the function $\mathrm{A}(x) = \int_0^x f(t)dt$. Find $\mathrm{A}(0)$, $\mathrm{A}(2)$, $\mathrm{A}(3)$, $\mathrm{A}(4)$, and $\mathrm{A}(5)$.



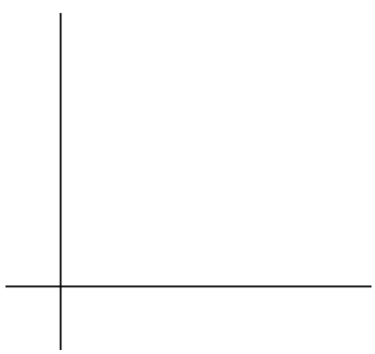
Example 2: Sketch the values of A in the graph below. Explain why the graph looks the way it does.



The Intermediate Value Theorem for continuous functions tells us that if $y = f(x)$ is continuous on a closed interval $[a, b]$, then it takes on every value between $f(a)$ and $f(b)$. If y_n is any number between $f(a)$ and $f(b)$, then y_n is equal to $f(c)$ for at least one value $x = c$ in which c is in the interval $x = [a, b]$. In other words, this means that a function which continuously traverses an interval in x or y must pass through every value in either interval on its way through.



The Mean Value Theorem tells us that if $y = f(x)$ is continuous at every point on the closed interval $[a, b]$ and differentiable at every point between a and b, then there is at least one number c such that $f'(c)$ is equal to $\frac{f(b)-f(a)}{b-a}$. Graphically, this means that if a line is drawn connecting the points $(a, f(x))$ and $(b, f(x))$ marking the endpoints of the interval, then there is at least one value $x = c$ in the same interval in which the tangent line to the curve is parallel to the line connecting the endpoints.



This means the instantaneous change in f over $[a, b]$ at some point between a and b is equal to the average change over the entire interval.

Graphically, this means that if a line is drawn connecting the points $(a, f(x))$ and $(b, f(x))$ marking the endpoints of the interval, then there is at least one value $x = c$ in the same interval in which the tangent line to the curve is parallel to the line connecting the endpoints.

If we sketch the derivative of the graph found in Example 2 by estimating slopes of tangents, we get a graph like the one found in Example 1.

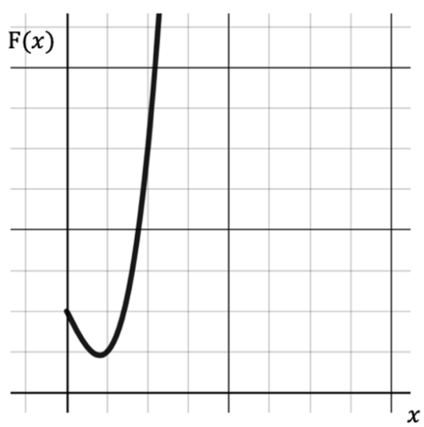
Differential calculus is related to a tangent problem and integral calculus is related to an area problem and the two are related to one another; they are both inverse processes. It was Isaac Barrow, an English Christian theologian and mathematician, who first realized this relationship. He did teach Newton after all!

Leibniz and Newton used this relationship to create a systematic method, and we now call this systematic method Calculus!

If f is continuous on $[a, b]$ and is the function $f(t)$, then we can define a new function on the definite integral and call it $\mathrm{F}(x) = \int_a^x f(t)dt$ in which x is in the interval $[a, b]$. It is another function of x that is the area under the curve between two points:

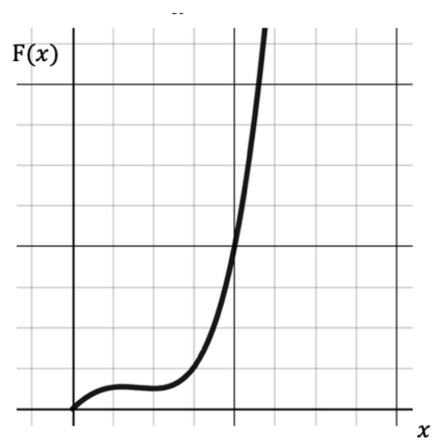
$$\frac{d\mathrm{F}}{dx} = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

The Fundamental Theorem of Calculus also tells us that if f is continuous at every point on $[a, b]$ and F is the antiderivative of f on $[a, b]$, then $\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a)$. Another way of saying this is to say that if a function f can be integrated and if $\mathrm{F}(x)$ is equal to $\int f(x)dx$, then the definite integral is $\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a)$.



$\mathrm{F}(x) = 4x^3 - 2x + 1$

Antiderivative



$f(x) = \frac{1}{4}x^4 - x^2 + x$

Integral Area

The antiderivative of $f'(x)$ is $f(x)$, but the antiderivative of $f(x)$ is $\mathrm{F}(x)$. Let L_n be the lower sums and U_n be the upper sums of an interval $[a, b]$ partitioned into n sub-intervals with the width of each sub-interval equal to Δx.

$$\lim_{n\to\infty} \mathrm{L}_n = \int_a^b f(x)dx$$

$$\lim_{n\to\infty} \mathrm{U}_n = \int_a^b f(x)dx$$

Let R_n be the Riemann sum that is equal to $\mathrm{F}(b) - \mathrm{F}(a)$. We saw from the previous examples that this is the exact value of the definite integral. The Fundamental Theorem of Calculus tells us that we can calculate the definite integral by evaluating the antiderivative at the upper limit of integration, U_n, and then subtracting from it the value of the antiderivative of the lower limit of integration, L_n, in which n is the number of intervals and does not affect the outcome.

The Riemann sum is between the lower sum and the upper sum.

$$\mathrm{L}_n \leq \mathrm{R}_n \leq \mathrm{U}_n$$

This means $\int_a^b f(x)dx \leq \mathrm{R}_n \leq \int_a^b f(x)dx$.

Because the lower sum and the upper sum are equal, the Riemann sum must be equal, $\int_a^b f(x)dx$ is equal to $\mathrm{F}(b) - \mathrm{F}(a)$, and $\mathrm{F}(b) - \mathrm{F}(a)$ is equal to $\int_a^b f(x)dx$.

<u>Looking Ahead 8.10</u>

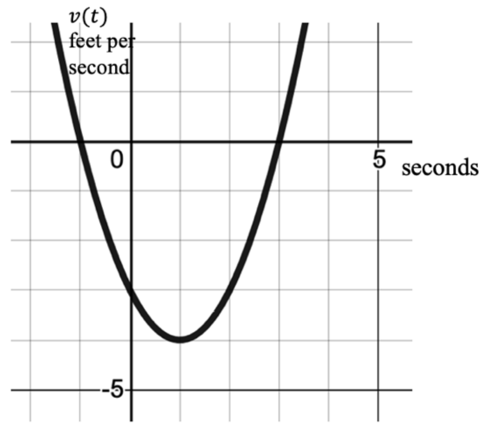
To evaluate a definite integral, follow the steps shown below:

a) Rewrite it as an indefinite integral
b) Evaluate the indefinite integral by finding the antiderivative
c) Apply the Fundamental Theorem of Calculus to it using subtraction

Example 3:	Evaluate $\int_1^3 2x^3 dx$.

Example 4: The velocity of a moving object in feet per second as a function of time t has equation $v(t) = (t+1)(t-3)$. Its graph is shown below. Use this information and the graph below to answer the given questions.

$$v(t) = t^2 - 2t - 3$$

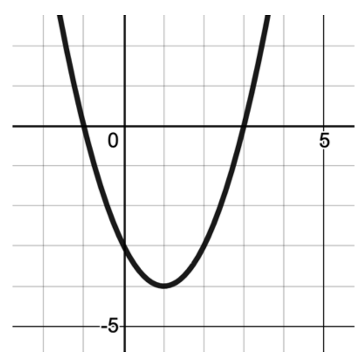


a) What is the velocity of the moving object at 2 seconds?

b) What is the acceleration of the moving object at 2 seconds?

c) Is the object speeding up or slowing down at 2 seconds?

d) What is the net displacement of the object from 1 to 3 seconds?



Example 5: Use the velocity function of the object from Example 4 to find the average velocity of the object over the time interval $[1, 3]$.

$$v(t) = t^2 - 2t - 3$$

The average velocity is $\frac{\Delta d}{\Delta t}$. The velocity function is the derivative of the distance function. The distance function is the area under the velocity curve; it is the antiderivative or integral of the velocity function.

$$d(t) = \frac{1}{3}t^3 - t^2 - 3t$$

The average velocity is the slope of the secant line between the two points where the graph of the function intersects the two endpoints of the time interval.

If we wanted to find the distance traveled between $t = 1$ and $t = 3$, it is the absolute value of -5.3 and is equal to 5.3 feet.

Distance is the interval between two locations. Because it is a scalar quantity, it has magnitude only. Displacement is also the interval between two locations, but it is a vector quantity and has both magnitude and direction.

The distance is the measurement of the actual path taken between two points. Displacement can be equal to or less than the distance, but it can never be greater than the distance. Therefore, distance can never be less than the displacement. It is always greater than or equal to the displacement.

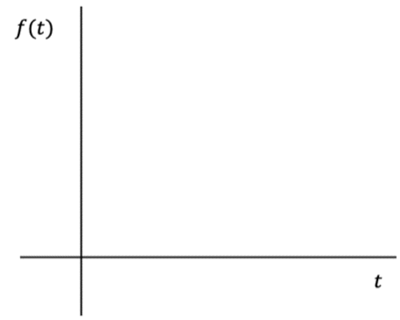
$$\text{Distance} \geq \text{Displacement}$$
$$\text{Displacement} \leq \text{Distance}$$

Section 8.11 Antiderivatives and the Indefinite Integral

Looking Back 8.11

If $\mathrm{A}(x)$ represents a function that determines the area under a curve between $[0, x]$ in which 0 is a fixed point and x is an unknown variable, then $\int_c^x f(t)dt$ is equal to $\mathrm{A}(x) - \mathrm{A}(c)$ in which $\mathrm{A}(c)$ represents a constant. We will call it C in which $\mathrm{A}(x)$ represents the area from 0 to x and $\mathrm{A}(c)$ represents the area from 0 to c. Subtracting $\mathrm{A}(c)$ from $\mathrm{A}(x)$ results in the area from c to x.

Therefore, $\int_c^x f(t)dt$ is equal to $\mathrm{A}(x) + \mathrm{C}$ because $\mathrm{A}(c)$ is a constant. Because C represents any constant, we will call it an arbitrary constant. As was shown in the previous Practice Problems section, picking any lower bound for c simply changes the two area functions by a constant. The general function $\int f(x)dx = \mathrm{A}(x) + \mathrm{C}$ is called the indefinite integral. If the function $f(x)$ is a derivative, the symbol $\int$ is the integral sign, f is the integrand, and x is the variable of integration.



We have been using $\int f(t)dt$ and $\int f(x)dx$. It does not matter which variable is used to represent x as long as it is consistent throughout the integral. Both represent an unknown variable.

The indefinite integral is related to the definite integral, but there are also distinct differences between them. One major difference is that the indefinite integral is a function such as $\int (3x - 4)dx$ or $\int (3t - 4)dt$ whereas the definite integral has boundaries that are defined such as in a function $\int_1^7 (3x - 4)dx$ or $\int_1^7 (3t - 4)dt$.

Looking Ahead 8.11

Example 1:	Find the derivatives for the functions below.

a) $\frac{d}{dx}[x^3]$

b) $\frac{d}{dx}[x^3 - 1]$

c) $\frac{d}{dx}[x^3 + 2\pi]$

d) $\frac{d}{dx}[x^3 \pm \mathrm{C}]$ (in which C is any constant)

Solving these gives us a little more insight into the general function of the indefinite integral:

$$\int f(x)dx = \mathrm{A}(x) + \mathrm{C}$$

The derivative of any constant with respect to x is 0, so all of the derivatives are the same.

When we are looking for the antiderivative, the question we are really asking is, "Given any expression, what is it the derivative of?" In this example, we are looking for the antiderivative of $3x^2$, which could be x^3, $x^3 - 1$, $x^3 + 2\pi$, or $x^3 \pm \mathrm{C}$. Because subtraction can be written as addition, we will make $x^3 \pm \mathrm{C}$ simply $x^3 + \mathrm{C}$.

We use the integral notation for this: Finding $\int 3x^2 dx$, the indefinite integral of the integrand really means to find the antiderivative of $3x^2$, which is in general $x^3 + \mathrm{C}$. As we have already learned, $\int 3x^2 dx$ is the indefinite integral of the integrand $3x^2$. Now we are calling it the antiderivative of $3x^2$, which is x^3 plus any constant $(x^3 + \mathrm{C})$. It is easy to forget the undetermined constant C, but neglecting it reduces the generality of the antiderivative. The antiderivative is a family of functions related by the constant C.

When we learned about derivatives, we learned the power rule: $\frac{d}{dx}[x^n] = nx^{n-1}$.

Example 2:	Evaluate $\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] + \mathrm{C}$ using the power rule and the sum rule.

Example 3:	Using the solution for Example 2, what is the antiderivative of x^n? Evaluate $\int x^n dx$.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + \mathrm{C}$$

This applies for any value n given $n \neq -1$. Again, we will call this an application of the reverse power rule. To sum it up, if the derivative of x^n is nx^{n-1}, then the antiderivative of nx^{n-1} is $x^n + \mathrm{C}$. The antiderivative of x^n is $\frac{x^{n+1}}{n+1} + \mathrm{C}$, in which C is a constant.

Example 4:	Evaluate $\int x^4 dx$ and $\int 3x^4 dx$.

Example 5:	Find the antiderivative of $3x^3 - 2x^2 + 5x - 2$.

<u>Section 8.12 Rules of Integration</u>

<u>Looking Back 8.12</u>

There are rules for integration that make the process easier and more understandable. One rule we have learned that makes integration easier is shown below:

$$\int_a^a f(x)dx = 0 \text{ and } \int_b^b f(x)dx = 0$$

This rule is logical because there is no Δx from $x = a$ to $x = a$ or no area under the curve over the interval $[b, b]$. This can be written as shown below:

$$\mathrm{A}(x) = \mathrm{F}(a) - \mathrm{F}(a) \text{ or } \mathrm{A}(x) = \mathrm{F}(b) - \mathrm{F}(b)$$

In both cases, the integral is equal to 0. This leads us to another rule:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \text{ and } \mathrm{A}(x) = -\int_a^b f(x)dx \text{ when } f(x) \leq 0$$

This means $\mathrm{A}(x) = \mathrm{F}(b) - \mathrm{F}(a)$ or $\mathrm{A}(x) = -[\mathrm{F}(a) - \mathrm{F}(b)]$, which can be written as shown below:

$$-\mathrm{F}(a) + \mathrm{F}(b) = \mathrm{F}(b) - \mathrm{F}(a)$$

Another rule we have learned to make integration easier is that the constant in front of a function is a scalar and can be moved in front of the integral sign:

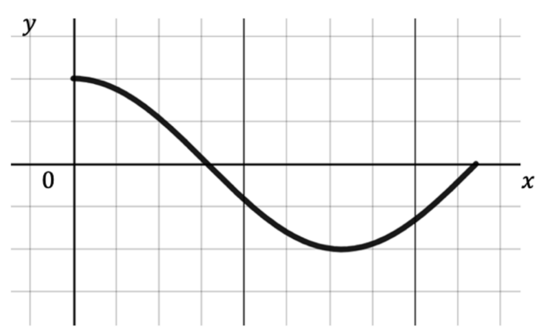
$$\int_a^b c\, f(x)dx = c\int_a^b f(x)dx \text{ (given } c \text{ is any constant)}$$

This leads to another rule:

$$\int_a^b c\ dx = c(b - a) \text{ (given } c \text{ is a constant on the interval } [a, b])$$

Let us look at $\int_0^{\frac{\pi}{2}} \cos dx$. Evaluate the integral:

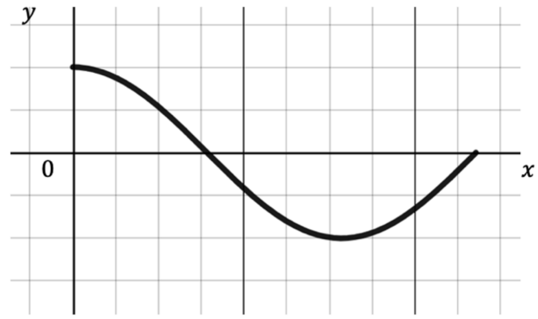
$$\sin\Big|_0^{\frac{\pi}{2}} = \sin\frac{\pi}{2} - \sin 0 = 1 - 0 = 1$$



It makes sense that the solution is a positive 1. The integral is positive, the integrand is positive, Δx is positive, and $f(x)$ is positive in the interval.

Now, let us look at $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx$. Evaluate the integral:

$$\sin \Big|_{\frac{\pi}{2}}^{\pi} = \sin \pi - \sin \frac{\pi}{2} = 0 - 1 = -1$$



This does not seem to make sense if we just look at the positive integral, the positive integrand, and the positive Δx; however, $f(x)$ is negative over the given interval in this scenario. We can see why we get -1 when we investigate this graph.

<u>Looking Ahead 8.12</u>

Let us see if we can come up with a rule for integrating a function when it is being added to another integral with the same integrand.

Example 1: Evaluate $\int_0^3 (2x^2 + 3)dx$ and $\int_3^5 (2x^2 + 3)dx$.

Example 2: Now evaluate $\int_{0}^{5} (2x^2 + 3)dx$ and make a conjecture based on your results and the results of Example 1.

A general rule for the sum of the integrals with the same integrand is as follows:

$$\int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx = \int_{a}^{c} f(x)dx$$

This is the Interval Addition Rule. Let us investigate the sum of integrals if the functions do not have the same integrand. The difference rule tells us that $\int [f(x) - g(x)]dx$ is equal to $\int f(x)dx - \int g(x)dx$.

Example 3: Evaluate $\int [(2\cos x) - (3\sin x)]dx$.

We need C for our undetermined constant at the end so this is $2\sin(\mathrm{x}) + 3\cos(\mathrm{x}) + \mathrm{C}$.

Notice that $\int \cos y \, dy$ is equal to $\sin x$.

The differential with respect to $y, dy,$ identifies the variable of integration. Any letter can be used to represent this variable (dt, dr, or dx).

$$\int \cos t \, dt = \sin x$$
$$\int \cos r \, dr = \sin x$$

The argument, dt, dr, dx, must match the variable of the function because it is the differential of the integrand. If the integrand function does not refer to the variable indicated by the differential, then it is a constant relative to this variable, and the earlier rule for integrating a constant applies:

$$\int_{a}^{b} f(x)dt = f(x) \cdot (b - a)$$

Example 4: Evaluate the composite function $\int 3\sin(3x+1)dx$.

Example 5: Evaluate the composite function $\int \sin(3x+1)\,dx$.

Section 8.13 Using Technology to Evaluate Integrals

Looking Back 8.13

Just as we used technology to check our answers for the Riemann sum, we can also use technology to check our answers for integral problems.

Example 1:	Use the calculator to find $\int \sin(x)\, dx$.

We can check the integral by finding the first derivative of the solution.

Use your calculator to find $\frac{d}{dx}(-\cos(x))$.

We recently learned how to find the derivative of the integral.

Example 2:	Use the calculator to find $\frac{d}{dx}\left(\int \sin(x)\,dx\right)$.

If you forget what any of these symbols mean or where the solutions come from, review the sections that discuss derivatives, definite integrals, antiderivative, and the indefinite integral (Section 8.3, Section 8.9, and Section 8.11).

Looking Ahead 8.13

We can check the amount of ingredients needed for the Streusel Mug Cake from Section 8.10 by using the key for the limit. We can also go to the menu of the calculator page, scroll down to the command for Calculus and click on the derivative or limit. The integral from this menu is the definite integral. We can find the indefinite integral symbol using the catalog key.

Section 8.14 Derivatives and Integrals

Looking Back 8.14

Rather than a module review, we are going to do some practice with derivatives and integrals and the methods we have learned to solve them, namely differentiation and integration.

In the next section, we will explore how to find the volume of three-dimensional solids called solids of revolution and conclude the module with a project. This will summarize all we have learned about derivatives and integrals and the powers of Calculus.

From the Fundamental Theorem of Calculus, we know that because $f(x) = \frac{1}{x}$ is continuous except at $x = 0$ when a and x are positive numbers, that it has an antiderivative $\mathrm{F}(x) = \int_a^x \frac{1}{t}\,dt$. We will see that this function, $\mathrm{F}(x)$, is the natural logarithm function, $\ln x$. If you graph the indefinite integral $\int \frac{1}{x}\,dx$ and graph $\ln x$, the results are the same graph.

The Natural Logarithm Function, $\ln x$, is equal to $\int_1^x \frac{1}{t}\,dt$ for $x > 0$, and $\frac{d}{dx}\ln x$ is equal to $\frac{d}{dx}\int_1^x \frac{1}{t}\,dt$ is equal to $\frac{1}{x}$ by the Fundamental Theorem of Calculus.

Notice that $\int \frac{1}{x}\,dx$ is the one function in which the power rule does not apply even though $\frac{1}{x}$ is equal to x^{-1}. The power rule would give $\frac{1}{-1+1}x^{1-1} = \frac{1}{0}$ as the antiderivative which clearly cannot be correct. For every positive value of x, $\frac{d}{dx}\ln x$ is equal to $\frac{1}{x}$. The chain rule can be applied in this scenario because all values are positive and $\ln u$ can be defined as follows:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

For the function $y = \ln u$, $\frac{d}{dx}\ln u$ is equal to $\frac{d}{du}\ln u \cdot \frac{du}{dx}$ is equal to $\frac{1}{u}\,\frac{du}{dx}$. Now, for any value $u > 0$, $\frac{d}{dx}\ln u$ is equal to $\frac{1}{u}\,\frac{du}{dx}$ and $\int \frac{1}{u}\,du$ is equal to $\ln u + c$.

The rules for logarithms apply to the natural logarithm.

Let a and x be positive numbers and n be any exponent.

$$\ln ax = \ln a + \ln x$$

$$\ln \frac{a}{x} = \ln a - \ln x$$

(When a is equal to 1, then $\ln \frac{1}{x}$ is equal to $-\ln x$ because $\ln 1$ is equal to 0.)

$$\ln x^n = n\ \ln x$$

In the examples that follow, we will prove what is stated in the start of this section.

Example 1: In Section 7 of Module 2, we learned that if x is equal to $\int_1^y \frac{1}{t}dt$ then y is equal to $e^x + c$. Evaluate the integral.

Example 2: Demonstrate that $\frac{d}{dx}[\ln x]$ is equal to $\frac{1}{x}$.

$$\therefore \frac{1}{x}\ln(\lim_{n\to 0}(1+n)^{\frac{1}{n}}) = \frac{1}{x}\ \ln e = \frac{1}{x}\cdot 1 = \frac{1}{x}$$

Previously, we have seen that the natural exponential function is its own derivative. Now, let us investigate why this is so through examples.

Example 3:	Demonstrate that $\frac{d}{dx}[e^x]$ is equal to e^x.

The derivatives of natural logarithms are as follows:

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)}f'(x)$$

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

Use these formulas to find the derivatives in the examples that follow.

Example 4:	Find $\frac{d}{dx}\ln(x+1)$.

Example 5:	Find $\frac{d}{dx}\ln x^4$.

Example 6: Use the chain rule with a clever form of 1 to evaluate $\int (2x+1)^2 dx$ and confirm it by finding $\int (4x^2+4x+1)dx$.

Example 7: Use the product rule to find $\frac{dy}{dx}$ if y is equal to $e^{2x} \ln x$.

Integration by parts comes from the product rule when y is equal to uv and u and v are differentiable functions of x.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

In differential form, this is written as follows:

$$d(uv) = udv + vdu$$

This can be rearranged as follows:

$$udv = d(uv) - vdu$$

This can also be written in indefinite integral form because $\int d\ (uv) = \int d(y) = y$ and $y = uv$:

$$\int udv = uv - \int vdu$$

The first integral, $\int udv$, is expressed in terms of the second interval, $\int vdu$.

Example 8:	Evaluate the definite integral using integration by part: $$\int_{-1}^{3} xe^{-x}\,dx$$

There is one other type of differentiation that we have investigated, and it is called implicit differentiation. Some graphs are differentiable in parts but not as a whole because they are continuous in some parts and not continuous in other parts.

Implicit differentiation defines y as a differentiable function of x, $\frac{dy}{dx}$. We treat y as differentiable but it is otherwise unknown. Simply differentiate both sides of the equation and then solve for $\frac{dy}{dx}$.

In the circle $x^2 + y^2 = 4$, the top half is represented by the equation $f(x) = \sqrt{4 - x^2}$ and the bottom half is represented by the equation $g(x) = -\sqrt{4 - x^2}$. These are differentiable except at $x = 2$ and $x = -2$. This equation does not let us define y explicitly in terms of x when $\mathrm{F}(x, y) = 0$ and there will be a nonvertical tangent line at every point, except at $x = 2$ and $x = -2$. We can use implicit differentiation to find $\frac{dy}{dx}$.

$$x^2 + y^2 = 4$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(4)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 9:	Find $\frac{dy}{dx}$ for the equation $3y = x^3 + \cos y$. Use implicit differentiation.

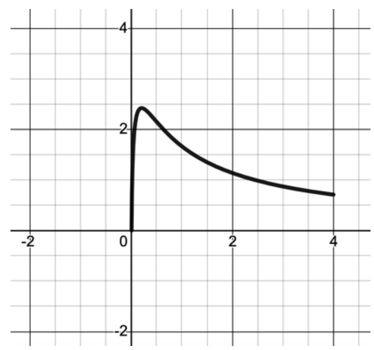
We said when we studied limits of indeterminate form that they could be evaluated using derivatives. The rule that allows us to do this is called L' Hopital's Rule and is named after G.F.A. L' Hopital (1661-1704).
When rational functions are of indeterminate form, then the limit may be found using the derivative of the numerator and the derivative of the denominator. We will end this section with an example of this.

Example 10:	Find $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Section 8.15 Solids of Revolution

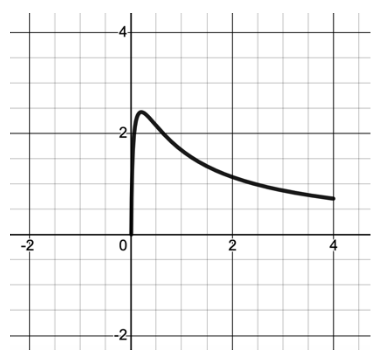
Looking Back 8.15

For a two-dimensional region, we used formulas from Geometry and extended lines vertically across the figure to find the area of each section. We added up the area of each section using Riemann sums to calculate the total area over an interval $[a, b]$.



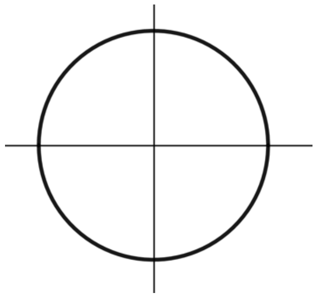
However, if we take the function $f(x)$ and rotate it about the x-axis, it creates a solid of revolution. We can use science to find the volume of this three-dimensional object by printing it on a 3-D printer and filling it with water. We can also use Calculus to calculate the exact volume of the object.

The function tells us the values of f at various values of x. The x-axis is called the axis of revolution. The revolution of the function around this axis creates circular disks sometimes called rings because of their circular shape.

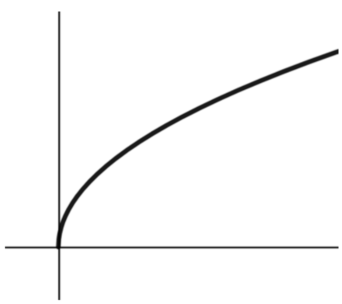


For a three-dimensional region, we can extend cross-sections of area across the figure and use the integral to add up all of these disks (rings) to get volume. The area of a circle is formed by the shaded region whose radius is the function $f(x)$. Therefore, $\mathrm{A} = \pi r^2$ becomes $\mathrm{A} = \pi[f(x)]^2$.

Example 1: Use Calculus to derive the formula for the volume of a sphere, $\mathrm{V} = \frac{4}{3}\pi r^3$, from the formula for the area of a circle, $\mathrm{A} = \pi r^2$, by rotating a hemisphere of the circle around the $x -$ axis.



Suppose you want to find the volume of a glass used for dessert pudding. You could place the glass along the x-axis and trace the shape of it; points on the function include: (1, 1), (2, 1.4), (3, 1.73), (4, 2), (5, 2.23), (6, 2.44), (7, 2.64), (8, 2.82), and (9, 3); then, using the graphing calculator to plot these points, you see that it models a square root function.



If we take $\int_0^1 \mathrm{A}(x)dx$ in which the function $\mathrm{A}(x)$ is the area, then $\mathrm{A} = \pi r^2$ becomes $\mathrm{A} = \pi(\sqrt{x})^2$ because r is equal to $\sqrt{x}$.

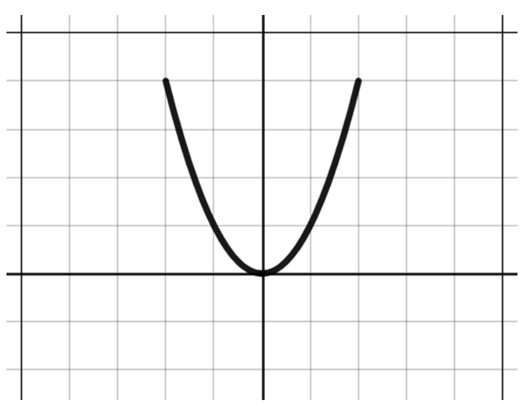
Calculating the volume of a solid of revolution is simple integration, $\int_a^b \mathrm{A}(x)dx$. We must be able to look at a shape and know the function of its area.

Example 2: Use integration to find the volume of the dessert cup over the interval $[0, 10]$. The units of measure are centimeters.

Example 3: Find the volume of a solid of revolution that results from rotating $y = x^2 + 4$ around the x-axis and over the interval $[0, 1]$.

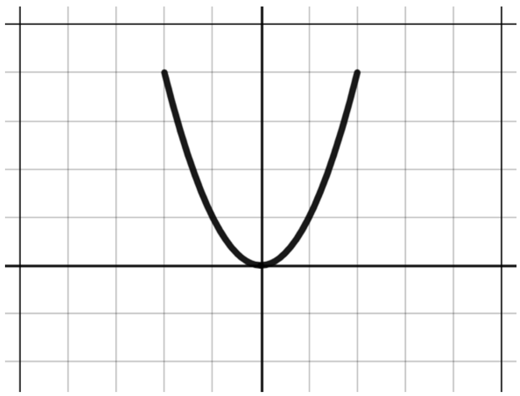
A washer is like a ring with a hole in the middle. If there is a hole in the center of revolution, then there is an outer radius and an inner radius. They both change and the inner must be subtracted from the outer to get the volume of the ring. There are two solids of revolution. If the outer function is $f(x)$ and the inner function is $g(x)$, the following equation is true:

$$\pi\int_a^b [f(x)]^2 dx - \pi\int_a^b [g(x)]^2 dx$$



Solids may also be formed by rotating the function around the y-axis. The depth is not dx, but dy. If the function is $y = x^2$, then x is equal to $\sqrt{y}$ and r is equal to x. Therefore, the radius is $\sqrt{y}$. The area is $A = \pi(\sqrt{y})^2 = \pi y$. The area of the surface multiplied by the depth (dy) is $\pi y dy$, which becomes the volume of the disk.

Example 4:	Find the volume of the disk over the interval $[1, 3]$ for $y = x^2$.



I would like to conclude this course by noting the collaborative works of Gottfried Wilhelm Leibniz, a German mathematician and philosopher, and Sir Isaac Newton, an English mathematician, physicist, astronomer, and theologian.

During their lives, there was much controversy over who invented Calculus. However, between 1699 and 1711, both men contributed greatly and probably independently to laying the groundwork for Calculus.

Leibniz was an optimistic rationalist. He believed that he lived in the best world God could have created. Sir Isaac Newton believed in one God who could not be denied due to His masterful works and the grandeur of creation. He felt the beautiful system of the sun and planets were governed by this Eternal Ruler.

Calculus is a beautiful system and a culmination and application of all the previous math we have learned. It has been my pleasure to teach you Pre-Calculus and Calculus and learn with you throughout the "Math with Mrs. Brown" courses. The Creator has allowed me the privilege of running the race with you and finishing the course.

I would also like to thank my son, Ethan, for his collaborative work in typing and editing these texts and uploading and editing the videos. Without him, I would have not been able to accomplish the dream God has given me.