Pre-Calculus and Calculus Module 7 Derivatives and Differentiation

Section 7.1 Experimental Rates of Change

Looking Back 7.1

Long ago, Tootsie Roll® Pops (TRP) Industries produced a commercial in which a wise Mr. Owl was asked: "How many licks does it take to get to the Tootsie Roll® center of a Tootsie Pop®?" (Watch the commercial to see Mr. Owl's response.)

In this section, we are going to perform a similar experiment to find rates of change. We will be analyzing the data from the experiment we perform to calculate the average value of the rate of change of the volume of a Tootsie Pop®.

The original experiment was conducted by a student who sucked on a Tootsie Pop® fairly uniformly for 30-second intervals until the chewy Tootsie Roll® center was reached. At the end of each 30-second interval, the student wrapped dental floss around the center of the Tootsie Pop® and laid it alongside a ruler to measure its circumference. From the measurement of the circumference the radius and volume could be calculated. A Tootsie Pop® is shaped like a sphere so the formula $V = \frac{4}{3}\pi r^3$ is used to find its volume.

The steps of the experiment will be reviewed in the Looking Ahead portion of this section. In the Practice Problems, you will conduct your own experiment.

Before we move on, we need to understand the tangent to a curve when the curve is a straight line. We have learned about average rates of change (AROC), instantaneous rates of change (IROC), and secant and tangent lines. We have also learned about limits. Now, let us talk about the tangent to a curve when the curve is a straight line.

We know that the IROC is the limit of a function as h approaches 0. For a linear function $y = mx + b$:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{m(x+h)+b-(mx+b)}{h} = \lim_{h\to 0}\frac{mh}{h}$$

Notice that at every x, the tangent to the curve when the curve is a straight line is the slope of the line. That makes sense because the rate of change is constant in a linear relationship.

Looking Ahead 7.1

The time and circumference from the Tootsie Pop® experiment are given below. We will complete the table as we continue through the steps to calculate the average value of the rate of change of the volume of a Tootsie Pop®.

Time (s.)	**Circumference c (cm.)**	**Radius r (cm.)**	**Volume V (cm.^{3})**	**ΔV (cm.^{3})**	$\frac{\Delta V}{\Delta t}$ $\left(\frac{\text{cm.}^3}{\text{s.}}\right)$
0	9.5				
30	9.2				
60	9.0				
90	8.8				
120	8.5				
150	8.3				
180	8.0				
210	7.7				
240	7.5				
270	7.2				
300	7.0				
330	6.7				
360	6.3				

1. Complete the radius column (**Radius r (cm.)**) of the table. (Divide the circumference by 2π and round to the nearest thousandth.)

2. Draw a time versus radius graph for the data. What do you expect this graph to look like?

3. Find the slope of the tangent line of the time versus radius graph. What does this slope represent?

4. Complete the volume column (**Volume V (cm.^{3})**) of the table. (Use the formula for volume of a sphere ($V = \frac{4}{3}\pi r^3$) and round to the nearest ten-thousandth.)

5. Draw a time versus volume graph for the data. What do you expect this graph to look like?

6. Complete the change in volume column ($\Delta \mathbf{V}$ ($\mathbf{cm.}^{\mathbf{3}}$)) of the table. (Subtract the volume at each 30-second interval from the previous 30-second interval of volume.)

7. Complete the change in volume over the change in time column ($\frac{\Delta \mathbf{V}}{\Delta \boldsymbol{t}}$ ($\frac{\mathbf{cm.}^{\mathbf{3}}}{\mathbf{s.}}$)) of the table. (Divide ΔV by Δt, which is 30 seconds for each interval, and round to the nearest ten-thousandth.)

8. Draw a time versus change in volume graph for the data. What do you expect this graph to look like?

9. Find the average of the change in volume over the change in time column ($\frac{\Delta V}{\Delta t}$ ($\frac{cm.^3}{s.}$)). (Add up all the $\frac{\Delta V}{\Delta t}$ values and divide by the total number of values, which is 12.)

10. What is the average value of the rate of change of the volume of the Tootsie Pop®? How does this compare to the slope of the time versus volume graph?

Section 7.2 Acceleration Versus Velocity Graphs

Looking Back 7.2

Previously, we learned that the instantaneous rate of change is called the derivative. It is called this because it is derived from the slope of the tangent line at a single point on the curve.

Velocity is the time rate of change of an object's position. Acceleration is the time rate of change of an object's velocity. Velocity is the calculated first derivative and acceleration is the calculated second derivative.

We have compared distance and velocity graphs in the past. The slope of the distance graph becomes the velocity graph. Now, we will investigate velocity and acceleration graphs in detail and see that the slope of the velocity graph becomes the acceleration graph.

Looking Ahead 7.2

Let us use an airplane takeoff problem to investigate displacement, velocity, and acceleration.

Example 1: Let us assume an airplane is taking off from a runway that is $100\ \text{m}$. Let us assume also that acceleration is constant. If the velocity at takeoff is $280\ \text{km./hr.}$, what is the acceleration at takeoff?

Three motion equations that are related to the equation used in Example 1 are shown as follows:

1) $v_f = v_i + at$ (in which v_i is initial velocity and v_f is final velocity, a is acceleration and t is time.

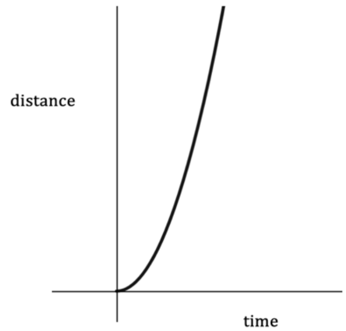
2) ${v_f}^2 = {v_i}^2 + 2ad$

3) $d = v_i t + \frac{1}{2}at^2$

Example 2: Dietrich travels at a constant velocity of $0.4\ \text{miles/minute}$ for 8 minutes. He then decelerates at $-0.2\ \text{miles/minute}^2$ for 2 minutes. Use equations to determine Dietrich's total distance traveled. Check your solution using a velocity versus time graph.

Remember that the area under the velocity curve represents the total distance.

Example 3: Let us revisit the function $f(x) = x^2$. How does the graph of $f'(x)$ relate to the graph of $f(x)$?



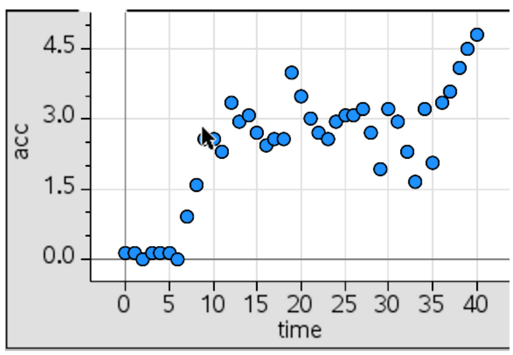
The first derivative of the time versus distance graph is the time versus velocity graph. The area under the time versus velocity graph is the displacement over the same time interval.

Example 4: Calculate the second derivative of $f(x) = x^2$. How does the graph of $f''(x)$ relate to the graph of $f(x)$?

The second derivative of the time versus distance graph is the time versus acceleration graph. The area under the time versus acceleration graph equals the change in velocity over the same time interval.

Example 5:	Find the average rate of change of the function $y = 2x^2 - 2$ over the interval $[-4, -2]$.

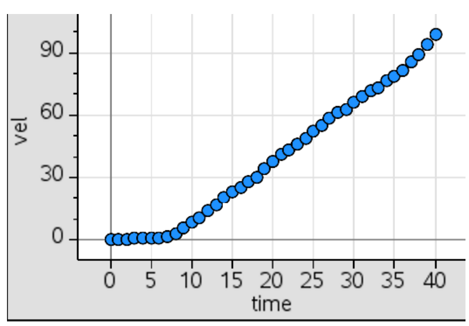
Let us look at the takeoff of an airplane again. This data is from a ____________________. The main motion is along the runway of an airport and acceleration is measured in meters per second squared. Let us look at a scatterplot of the acceleration versus time graph. This data is from Section 6.13 Practice Problems.



At the beginning, acceleration increases rapidly until it reaches about __________ $\mathrm{m/s^2}$ in which it remains constant up to takeoff (this is the point when it leaves the ground).

Because a is equal to $\frac{\Delta v}{\Delta t}$, we can solve for change in velocity by using the equation $\Delta v = a\Delta t$. Change in velocity is the acceleration multiplied by the length of the time interval, which is ____________________.

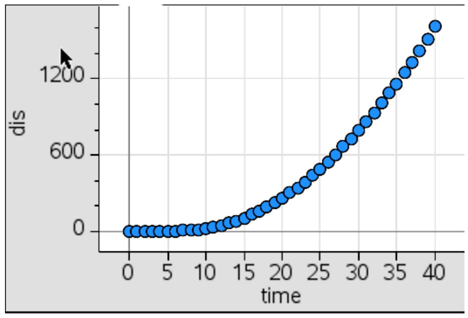
In order to get velocity as a function of time, we need to get a sum of the changes in velocity. This assumes velocity begins with 0. Now, we can graph time versus velocity.



The rise in velocity is almost linear. This corresponds to the constant portion of the acceleration graph. The break at the end of the graph is where the plane leaves the ground. The speed at takeoff is about __________ or __________. This speed is reached in less than 30 seconds.

Now, we can look at the distance versus time graph to see how far the plane went in the first 5 seconds of takeoff and how long the runway needs to be.

Because $v = \frac{\Delta s}{\Delta t}$ can be written "$\Delta s = v \cdot \Delta t$," we can follow the same steps we used previously to get a graph of displacement versus time. Because Δs is the change in position, we again need to get a sum of the changes in displacement for a cumulative distance.



Previously, we learned that takeoff was ___________ seconds after start. This means the plane traveled about _______________ in only 30 seconds.

We saw the shapes of these graphs in Section 6.13. Now, we can interpret what they mean.

Section 7.3 The Power Rule

Looking Back 7.3

God's universe is full of patterns. Perhaps we can find a pattern that makes finding derivatives easier. Moreover, perhaps we can find the instantaneous rate of change directly from the function instead of finding the average rate of change first.

Let us try to derive a formula for the instantaneous rate of change that can be used for any point c in the domain of a power function and bypass the whole limit process.

Looking Ahead 7.3

Example 1:	Find the average rate of change for $f(x) = 4x^3$ at $x = c$.

Example 2:	Use the four functions and their derivatives below to answer the questions that follow.

$$f(x) = x^2 \qquad f'(x) = 2x$$

$$f(x) = x^3 \qquad f'(x) = 3x^2$$

$$f(x) = x^5 \qquad f'(x) = 5x^4$$

$$f(x) = x^9 \qquad f'(x) = 9x^8$$

a) What do you notice about the change in coefficients?

b) What do you notice about the change in exponents?

Example 3:	Use the four functions and their derivatives below to answer the questions that follow.

$$f(x) = 2x^4 \qquad f'(x) = 8x^3$$

$$f(x) = 5x^3 \qquad f'(x) = 15x^2$$

$$f(x) = 4x^2 \qquad f'(x) = 8x$$

$$f(x) = -5x \qquad f'(x) = -5$$

a) Do you notice the same patterns with the coefficients as you did in Example 2?

b) Do you notice the same patterns with the exponents as you did in Example 2?

Example 4: Look at the four functions and their derivatives below. Do these follow the same patterns as in the previous functions?

$$f(x) = \frac{1}{x} \qquad f'(x) = -\frac{1}{x^2}$$

$$f(x) = \frac{2}{x^2} \qquad f'(x) = -\frac{4}{x^3}$$

$$f(x) = -\frac{1}{x^3} \qquad f'(x) = \frac{3}{x^4}$$

$$f(x) = \frac{3}{x^4} \qquad f'(x) = -\frac{12}{x^5}$$

These patterns in the functions and their derivatives are not as evident as in the previous examples. Let us put some intermediate steps in and use the rules of multiplying the exponent by the coefficient and making the exponent one less to see if they do.

$$f(x) = \frac{1}{x} \qquad f(x) = x^{-1} \qquad f'(x) = -1x^{-2} \qquad f'(x) = -\frac{1}{x^2}$$

$$f(x) = \frac{2}{x^2} \qquad f(x) = 2x^{-2} \qquad f'(x) = -4x^{-3} \qquad f'(x) = -\frac{4}{x^3}$$

$$f(x) = -\frac{1}{x^3} \qquad f(x) = -x^{-3} \qquad f'(x) = 3x^{-4} \qquad f'(x) = \frac{3}{x^4}$$

$$f(x) = \frac{3}{x^4} \qquad f(x) = 3x^{-4} \qquad f'(x) = -12x^{-5} \qquad f'(x) = -\frac{12}{x^5}$$

Let us look back at Example 1 to see if these functions follow a similar pattern to the ones in the previous examples.

$$f(x) = 4x^3 \qquad f'(x) = 4 \cdot 3x^2 \qquad f'(x) = 12x^2$$

The exponent does get multiplied by the coefficient of the original function for the derivative. The exponent does become one less. This is called the Power Rule. The dilation factor of 4 actually stays the same but gets multiplied by the coefficient from the exponent.

Example 4 does follow the same patterns as Example 1. It is called the Power Rule.

The Power Rule for the derivative states:

'If $f(x) = kx^n$ in which k and n are both real numbers, then $f'(x)$ is equal to $k \cdot nx^{n-1}$.'

In the Practice Problems section, we will use the Power Rule to find the derivative of a function when it is a monomial term. In the next section, we will find the derivative by using the Power Rule when the function is a polynomial. (You were invited to look for patterns for both of these in the previous Practice Problems section.)

Section 7.4 The Sum Rule

Looking Back 7.4

The foundations of Calculus were laid by Gottfried Wilhelm Leibniz (1646-1716) of Germany and Sir Isaac Newton (1643-1727) of England.

Gottfried Wilhelm Leibniz is considered the author of the Power Rule. He lived in the seventeenth and eighteenth century and was founding Calculus around the same time Sir Isaac Newton was founding Calculus. Because there is debate over who discovered what and when, both are considered to be co-founders of Calculus, although Leibniz is considered the last of the "universal geniuses." Denis Diderot, a French philosopher who was awed by Leibniz' works, wrote in *Oeuvres Complètes* (Vol. 7, p. 709) that what Leibniz "composed on the world, God, nature, and the soul is of the most sublime eloquence."

These high praises are the result of years of hard work and extensive reading from Gottfried Wilhelm Leibniz. When he was just six years old his father died, and he was given freedom over his extensive library. Therefore, he began his life by reading volumes on ancient history and the church fathers. Later, Leibniz spent four years in Paris in which he was tutored in philosophy, physics, and mathematics, and read the unpublished manuscripts of Descartes and Pascal. Eventually, all of this reading and studying led him to his work on Differential Calculus and the Infinite Series. Leibniz' work with these is what makes texts like this possible.

Despite all his contributions, Leibniz' writings and work were still overshadowed during his life when he was accused of stealing his ideas from Sir Isaac Newton. Now, it is widely believed that these two men developed their ideas independently. The Power Rule, however, is fully credited to Leibniz.

Previously, we have learned that if $y(t)$ is equal to t^n, then $y'(t)$ is equal to nt^{n-1}. Another way to write "f'" (Notation used by Newton) is "$\frac{dy}{dt}$" (Notation used by Leibniz) and $\frac{dy}{dt}$ is the derivative of y with respect to t, or the change in y over the change in t. When the function is the polynomial $y(t) = y_1(t) + y_2(t)$, then the derivative is $y'(t) = \frac{dy_1}{dt} + \frac{dy_2}{dt}$. This can be called the Sum Rule.

Looking Ahead 7.4

Example 1:	Find the derivative of the function $y(t) = 3t^2 - 2t$.

Example 2:	Find the derivative of the function $y(t) = -5t^3 + 8$.

Example 3:	Find the derivative of the function $y(t) = -12t^3 + 2t^2 - 4t + 0.5$.

In summary, if $p(x)$ is a polynomial function, then the derivative is the sum of the derivative of each of the monomial terms.

If $p(x)$ is equal to $f(x) + g(x) + h(x)$, etc., then $p'(x)$ is equal to $f'(x) + g'(x) + h'(x)$, etc.

If a polynomial is of degree 1, then it is a linear function and the derivative is a constant term, which is the slope of the line.

If a function is a constant only, the derivative is 0.

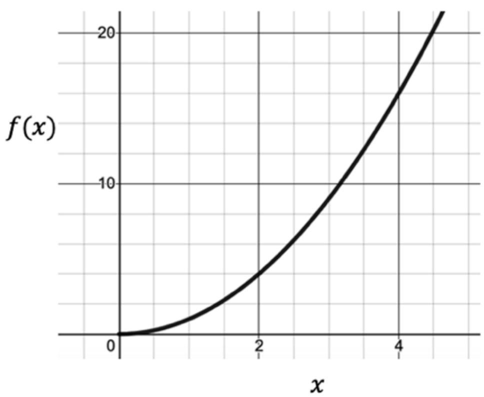
Section 7.5 Finding the Derivative Graphically

Looking Back 7.5

In the previous module, we learned that when $f(x)$ is equal to x^2, then $f'(x)$ is equal to $2x$. The function was quadratic, and the derivative was linear. The derivative is found graphically by finding the slope along the curve.

x	$f(x)$
0	0
1	1
2	4
3	9
4	16
5	25

Given the table for $f(x) = x^2$, the second difference is the constant 2, which is the slope of the derivative function. The first common difference is the slope of the secant line between consecutive points.



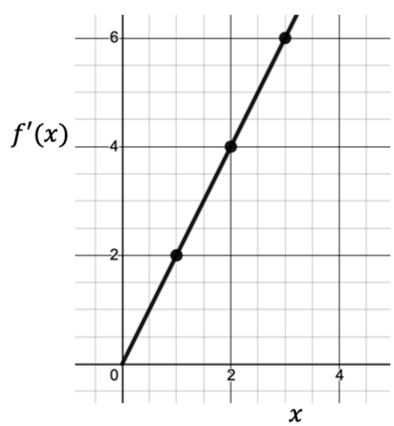
When x is equal to 1, the slope of $f(x)$ looks to be about 2.

When x is equal to 2, the slope of $f(x)$ looks to be about 4.

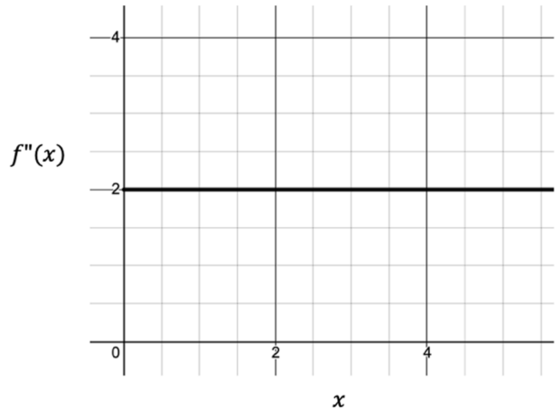
When x is equal to 3, the slope of $f(x)$ looks to be about 6.

If we plot the points (x_1, m_1), (x_2, m_2), (x_3, m_3), which are $(1, 2)$, $(2, 4)$, $(3, 6)$, it results in the following graph:

x	$f'(x)$
0	0
1	2
2	4
3	6
4	8



The curve is now a straight line. The slope is 2 all along the curve. The first common difference is 2. The second derivative is $f"(x) = 2$. The slope is constant so the graph for $f"(x)$ is a horizontal line.



Let us review concavity before proceeding with our examples. By now, we have seen that the derivative $f'(x)$, graphically speaking, tells you the instantaneous slope of the function $f(x)$ at every point where f' is defined, which means it gives you the slope of the tangent line to f at every point in question.

When $f"(x) > 0$, $f'(x)$, and the slope of $f(x)$ is increasing as x increases. The graph of $f(x)$ will curve upwards. This is concave up. If $f'(x)$>0, then the slope is positive, and $f(x)$ will curve upwards by becoming steeper, whereas if the slope is negative, $f'(x)$<0, then it will curve upwards by becoming less steep.

On the other hand, when $f"(x) < 0$, $f'(x)$, and the slope of $f(x)$ is decreasing as x increases. The graph of $f(x)$ will curve downwards. This is concave down. If $f'(x) < 0$, then the slope is negative, and $f(x)$will curve downwards by becoming steeper, whereas if the slope is positive, $f'(x) > 0$, then it will curve downwards by becoming less steep.

	$f'(x) > 0$ f is increasing	$f'(x) < 0$ f is decreasing	Concavity
$f"(x) > 0$ f curved upward			
$f"(x) < 0$ f curved downward			

The graphical property of upward or downward is linked to $f"(x) > 0$ and is called concavity. An upward curve is said to have positive concavity, and a downward curve is said to have negative concavity. Points where the concavity changes from one to the other are called inflection points. These always coincide with the value of x.

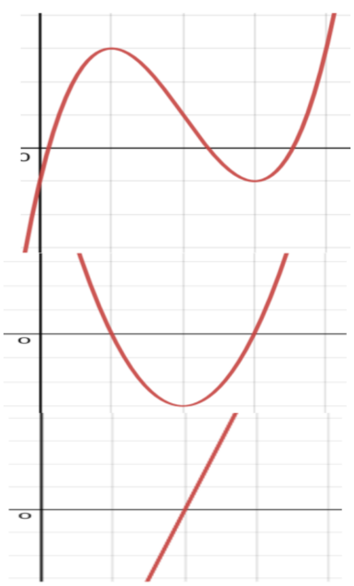
Examples of Inflection Points

Three Infection Points	No Inflection Points
Inflection Point where $f"(x) = 0$ also	No Inflection Points, but $f"(x) = 0$ at all points

Looking Ahead 7.5

Let us investigate a function and its first and second derivative aligned vertically.

Example 1: Use the given graph and its' derivatives to answer the questions that follow.



a) Where does $f(x)$ have an inflection point? To what does the point correspond with on the graphs of $f'(x)$ and $f''(x)$?

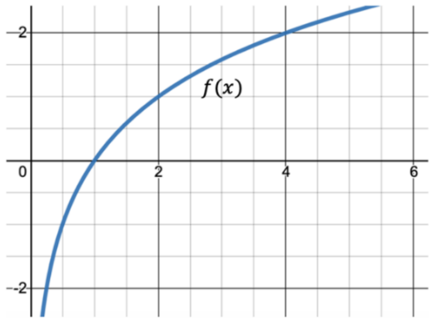
b) Where does $f(x)$ have a local minima or maxima? To what does the point correspond with on the graph of $f'(x)$, and how does it compare to $f''(x)$?

c) Do the zeros of $f(x)$ correspond to any points of significance on $f'(x)$ or $f''(x)$?

d) The graph of $f'(x)$ is negative between $x = 1$ and $x = 3$? What can you say about the graph of $f(x)$ over this same interval?

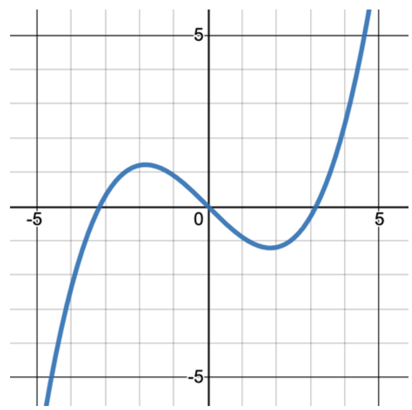
e) The graph of $f''(x)$ is negative to the left $x = 2$ and positive to the right of it. What bearing does this have on the graphs of $f(x)$ and $f'(x)$

Example 2: Given the graph of $f(x)$ shown below in blue, estimate the slope when x is equal to 0, 1, 2, 3, and 4, and then sketch the first derivative $f'(x)$ in red.



Example 3: Given the function $g(x)$ shown below in blue, estimate the slope when x is equal to -3, -2, -1, 0, 1, 2, and 3 to complete the table, and then sketch the first derivative $g'(x)$ in red.

x	**Slope at x on $g(x)$**
-3	
-2	
-1	
0	
1	
2	
3	



The slope is 0 at the local maxima and local minima when $x = -1.8$ and $x = 1.8$ respectively. Graphing $(x, \text{slope of } g(x) \text{ at } x)$ results in the graph of the derivative $g'(x)$ as $g'(x)$ gives the slope of $g(x)$ at x.

The slope is positive and decreasing over the interval $-\infty < x < -1.8$ for $g(x)$. So, the graph of $g'(x)$ is positive over that interval. Since the slope is 0 at the local maxima and local minima on $g(x)$, these are the x-intercepts.

The slope is negative over the interval $-1.8 < x < 1.8$ for $g(x)$ so $g'(x)$ is also negative over the interval, while negative throughout the interval, decreasing from $-1.8 < x < 0$ and increasing from $0 < x < 1.8$. From -1.8 to 0, the slope is negative and decreasing which means it is becoming steeper but downward. From 0 to 1.8, the slope is negative, but increasing so it is becoming less steep. This means concavity changed at 0 and 0 is an inflection point.

The slope is positive and increasing (getting steeper) over the interval $1.8 < x < \infty$ for $g(x)$ so it is positive and above the x-axis for $g'(x)$.

Section 7.6 The Derivatives of Sine and Cosine

Looking Back 7.6

The circular functions are often called the trigonometric functions. These include sine (sin), cosine (cos), tangent (tan), cosecant (csc), secant (sec), and cotangent (cot). There are a few methods to finding the derivatives of these functions.

One of these methods involves using implicit differentiation. This method can be used when the rules of differentiation (finding the derivative) are applied. We have learned two of these rules previously.

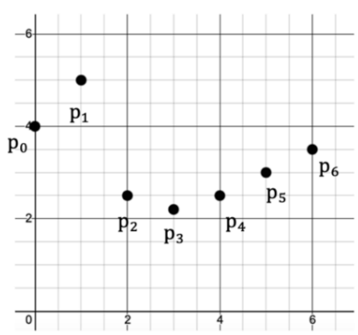
The Power Rule: If $f(x)$ is equal to kx^n in which n is a non-negative integer and k is a constant dilation factor, then $f'(x)$ is equal to $k \cdot nx^{n-1}$.

The Sum Rule: If $p(x)$ is equal to $f(x) + g(x) + h(x)$, then $p'(x)$ is equal to $f'(x) + g'(x) + h'(x)$. (This can also be called the Sum and Difference Rule.)

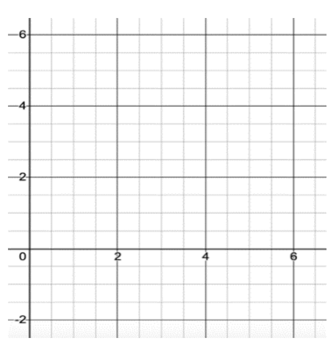
The method that involved estimating the slope at each of the various values of x on the original function and then plotting the slope points (x_1, m_1), which we learned in the previous section, is a method for finding the graph of the derivative function.

Looking Ahead 7.6

Example 1: Estimate the slope for the tangent to the curve at p_1, p_2, p_3, p_4, and p_5. Draw the graph of the derivative function.



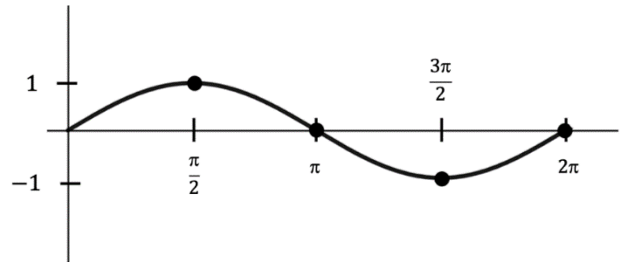
Note: Computing the slope of the secant line between p_1 and p_3 would be an approximation for the slope of the tangent line at p_2.

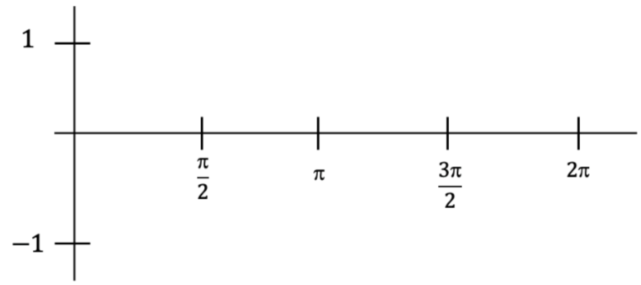


Example 2: Find $\frac{d^2y}{dx^2}$ for $y = 2x^3 + 3x^2 - 2x$.

Example 3: Let y be $x^3 + 4x$. Investigate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x < 0$, $x = 0$, and $x > 0$. Then let $g(x)$ be $-x^3 - 3x$ and investigate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x < 0$, $x = 0$, and $x > 0$. Explore the slope of the tangent lines and concavity.

Example 4: If $g(x)$ is equal to $\sin x$, then draw the graph of $g'(x)$ (the derivative of $\sin x$).





Connecting the points with a smooth curve results in the familiar cosine curve. The derivative of the sine function is the cosine function $g'(x) = \cos x$. In this Practice Problems section, we will be finding the derivative of the cosine function.

If there is a constant "a" and $f(x)$ is equal to $\sin(ax)$, then the derivative $f'(x)$ is equal to $a \cos ax$. Again, in Calculus notation, if the function is $y(t)$, then the derivative is $\frac{dy}{dt}$ (read "dy dt" or "the derivative of y with respect to t").

If $y(t)$ is equal to $\sin(at)$, then $\frac{dy}{dt}$ is equal to $a \cos at$.

Another way to write the derivative with respect to x is $\frac{d}{dx}$ (read "d dx").

$$\frac{d}{dx} \sin ax = a \cos ax$$

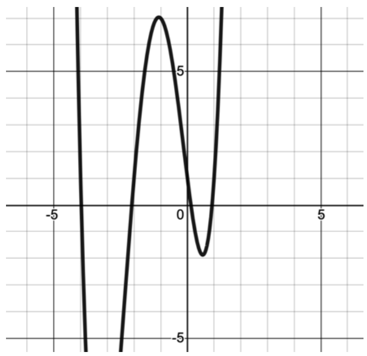
$$\frac{d}{dx} \sin x = \cos x$$

We will investigate the derivatives of the other circular functions after we learn the Product Rule, the Quotient Rule, and the Chain Rule for finding derivatives.

Section 7.7 Local and Global Extrema and Critical Points

Looking Back 7.7

In the functions we have studied, we have used the words "local maxima" and "local minima." These are different than global maxima and global minima. Global maxima or global minima (absolute maxima or absolute minima) are the highest or lowest values, respectively, that a given function can attain over its entire domain. A global extremum is therefore an upper or lower bound on the entire function: unbounded functions have no global extremum in the direction that they are unbounded.

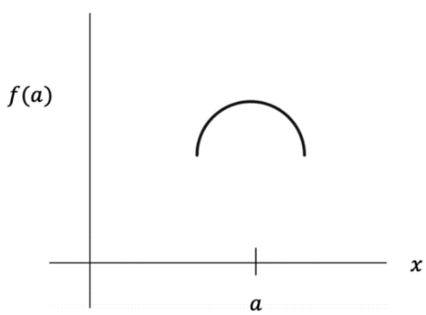


There is a local minimum at $x = 0.574$. It is a low point where the graph is decreasing and then begins increasing; it changes direction. There is a lower minimum at $x = -3.25$. In fact, it is the lowest point on the graph so it is considered a global minimum (absolute minimum).

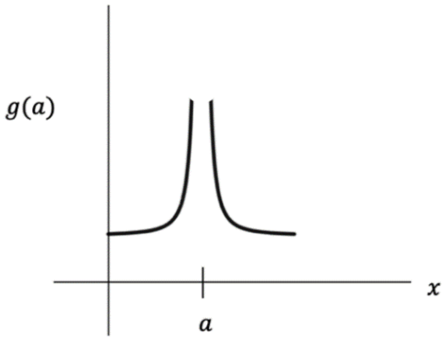
There is a local maximum at $x = -1.07$. The graph goes from increasing to decreasing; however, there is not a global maximum (absolute maximum) here as the end behavior of the function continues to increase.

Looking Ahead 7.7

Example 1: Given the graph of $f(x)$, describe and sketch $f'(x)$ and $f"(x)$. In particular, focus on the values/intervals when $x = a$, $x < a$, and $x > a$.



Example 2:	Given the graph of $g(x)$, describe $g'(x)$ and $g"(x)$ when $x = a$, $x < a$, and $x > a$.



The measure of the rate at which things change is called the derivative. Derivatives are defined as the limiting value of averages changes. We have been using the limiting values of the slopes of secants to derive the slopes of curves.

We can analyze graphs to find exact numerical derivatives. We have been using the slopes of tangent lines to do this.

In summary, the slope of a secant line is shown as follows:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

The slope of the graph of a point is the limit of the secant slope, which we define as the derivative, shown as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

A function is differentiable at a point if the point is in the domain of the derivative. A differentiable function is differentiable at every point in the domain. When the derivative of a point exists, the slope is tangent to the curve. Sometimes, a function is not continuous and does not have a limit. In this case, as in Example 2, the function is not differentiable.

There are some continuous functions that are not differentiable. If the right-sided and left-sided limits do not agree, then the right side of the tangent slope would not be equal to the left side of the tangent slope. In Example 2, $f"(a)$ does not exist at a. Again, that means that some functions are not differentiable.

Differentiability at a point:

A function is said to be differentiable at a point $x = a$ if the limit defining the derivative exists at that point. This means there are three conditions that must be met, some with sub-conditions:

1. $x = a$ must be in the domain of $f(x)$ so $f(a)$ must be defined and of finite value

2. $f(x)$ must be continuous at $x = a$ so $\lim_{x \to a} f(x) = f(a)$
   A) $\lim_{x \to a^-} f(x) = f(a)$ Left Continuous
   B) $\lim_{x \to a^+} f(x) = f(a)$ Right Continuous

3. $f(x)$ must be converge at $x = a$ so $\lim_{x \to a} [\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}] = f'(a)$
   A) $\lim_{x \to a^-} [\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}]$ must exist and be finite (Left Differentiable)
   B) $\lim_{x \to a^+} [\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}]$ must exist and be finite (Right Differentiable)
   C) $\lim_{x \to a^-} [\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}] = \lim_{x \to a^+} [\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}]$. The two must agree.

If all conditions are met except for 3C, then the function is said to be semi-differentiable at that point.

An example of this is an absolute value function, which is a non-differentiable, but continuous function.

If all conditions are met, then the function is differentiable at the point, and

$$f'(a) = [\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}]$$

The derivative is definable and unique.

Let us investigate some examples where a function is not differentiable at a point.

Condition 1 is not met and $x = a$ is not in the domain of $f(x)$.

Excluded Interval	Open Endpoint of Domain	Removable Discontinuity
Jump Discontinuity	Vertical Asymptote	Point where function is undefinable

Condition 1 is met, but Condition 2 is not met and $f(x)$ is defined but not continuous at $x = a$.

Jump Discontinuity	Removable Discontinuity	Closed Endpoint of a Domain	One side of a limit fails to exist	Both Sides of a limit fail to exist

Condition 1 and Condition 2 are met, but Condition 3 is not met and $f(x)$ is continuous but not smooth.

This could be a kink, corner or cusp where the right and left side limits defining the derivative exist but are not equal.

A vertical cusp

One or both of the one-sided limits defining the derivative is non-finite. If both are non-finite, then the infinite limits tend toward the opposite infinities.	A vertical tangent is when both one-sided limits tend toward the same infinity. This is a special case because though the vertical tangent line can be defined, the function is technically not definable at the point in question.

Example 3:	Let $h(x) = \frac{1}{4}x^4 + \frac{5}{3}x^3 + 3x^2$. Identify the local maxima and minima of $h(x)$.

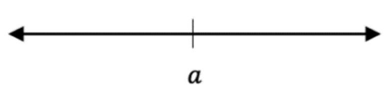
The x-intercepts of $h'(x)$ are critical points. Do you see how the first derivative changes at the critical points? Determine where it is positive and where it is negative to see if there is a local maximum or a local minimum. This is called the First Derivative Test for Local Extrema. Test points in between these critical points and to the left and right.

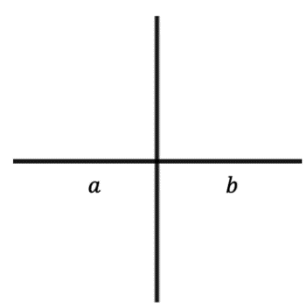
By observing the sign changes of the derivative at critical points, local maxima and minima can be determined for the function.

The second derivative also tells us if the critical points are local maxima or minima or neither. A function is concave up if the second derivative is positive because the critical point is a minimum. A function is concave down if the second derivative is negative because the critical point is a maximum. This is called the Second Derivative Test for Local Extrema, which we will explore further in the Practice Problems section.

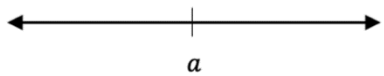
As in Example 2, we can have local extrema at a point in which the derivative of the function does not exist, such as $f'(a) = \text{DNE}$. This is because of the cusp.

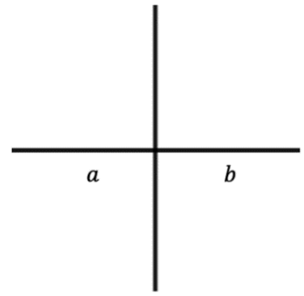
In summary, the First Derivative Test for Local Extrema states: "If the first derivative goes from negative to the left of a critical point to positive to the right of the critical point, then the point is a local minimum."



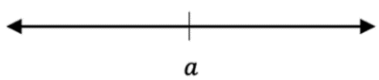


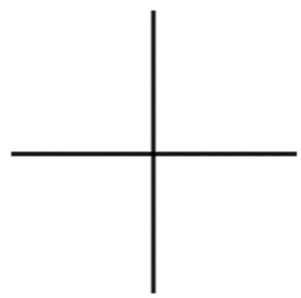
If the first derivative goes from positive to the left of a critical point to negative to the right of the critical point, then the point is a local maximum.



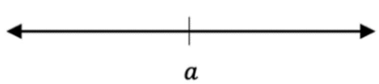


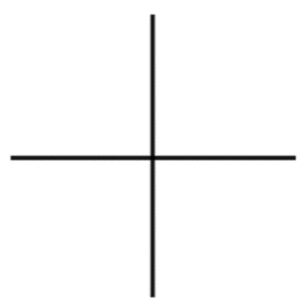
The Second Derivative for Local Extrema states: “If the first derivative is zero and the second derivative is negative at a critical point, then the point is a local maximum.”





If the first derivative is zero and the second derivative is positive at a critical point, then the point is a local minimum.





The Second Derivative Test for Local Extrema is easier to perform; however, it does not always apply when $f'(a)$ is equal to 0 and $f"(x)$ is equal to 0, but f has a minimum value at a, or f has a maximum value at a, or f has neither a maximum or minimum value at a. In this case, the First Derivative Test for Local Extrema may be applied instead.

Section 7.8 Finding the Equation of the Tangent Line

Looking Back 7.8

On the curve of any function, the tangent line makes contact with the curve at one point. The curve itself may be a quadratic, cubic, or exponential function, but the tangent line will be linear.

If a function is differentiable at $x = a$, then the unique linear function containing the point $(a, f(a))$ that also has the slope $f'(a)$. The line may serve as a close approximation for the graph of the function when x is close to a and is the tangent to the function at $x = a$. The slope of this tangent line is $m = \frac{f(x)-f(a)}{x-a}$, which results in $f(x) - f(a) = m(x - a)$. We know that the slope is the first derivative of the function and $m = f'(x)$. Substituting the first derivative for the slope gives us $f(x) - f(a) = f'(x)(x - a)$.

This linear function approximates the values of x close to a and is the equation of the tangent line. The derivative $\frac{dy}{dx}$ of a function $f(x)$ is the slope of the tangent line to that function at x.

We can also say that the line $t(x)$ which is tangent to the curve $f(x)$ at point $x = a$ must intersect the curve at that point and have the same slope as the curve at that point. Therefore,

$$m = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ and } t(a) - f(a),$$

$$\text{so } \frac{t(x)-f(a)}{x-a} = f'(a) \text{ or } t(x) = f'(a)x + \big(f(a) - a \cdot f'(a)\big)$$

Note that the slope of $t(x)$ is $f'(a)$, and the $y-$intercept is $f(a) - a \cdot f'(a)$.

Looking Ahead 7.8

Example 1: A line tangent to the graph of $y = f(x)$ at $x = 2$ passes through the points $(1, 4)$ and $(3, -2)$. Use this information to solve a)-c) below.

a) Find $f'(2)$.

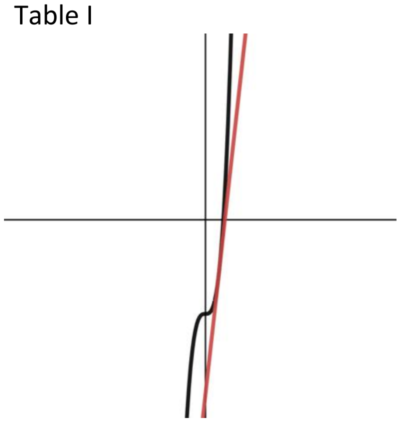
b) Find the equation of the line tangent to the function at $x = 2$.

c) Find $f(2)$.

Example 2: For $f(x) = 3x^3 - 7$, find the equation of the tangent line at the point $(1, -4)$.

Example 3: The graph of $f(x) = 3x^3 - 7$ and the tangent line $y = 9x - 13$ are shown below. Using the "Lists and Spreadsheets" page on the graphing calculator allows us to observe values of x that are close to a (which is 1 in this case). Call the tangent line $t(x)$ and find the amount of error between the function values $f(x)$ and the linear approximation at $t(x)$, which approximates the curve at that point. What do you notice about the error when $x = 1$? It vanishes! Use this information and the graph below to answer the questions that follow.

Table I



x	$f(x) = 3x^3 - 7$	$t(x) = 9x - 13$	Error $= t(x) - f(x)$
2	17	5	-12
1.1	-3.007	-3.1	-0.093
1.01	-3.909097	-3.91	-0.000903
1.001	-3.990990997	-3.991	-0.000009003
1	-4	-4	0

Table II

Δx	Δf	Δt	$\Delta t - \Delta x$
1	21	9	-12
0.1	0.993	0.9	-0.093
0.01	0.090903	0.09	-0.00093
0.001	0.009009003	0.009	-0.0000093
0	0	0	0

Table III

Δx	$\frac{\Delta f}{\Delta x}$	$\frac{\Delta t}{\Delta x}$	$\frac{\Delta t}{\Delta x} - \frac{\Delta f}{\Delta x}$
1	21	9	-12
0.1	9.93	9	-0.93
0.01	9.0903	9	-0.0903
0.001	9.009003	9	-0.009003
0	9	9	0

Answer the following questions.

a) Using Table I, as the point of tangency is approached, what function becomes a very good approximation for $f(x)$?

b) Using Table II, as the point of tangency is approached, what does $f(x)$ approach?

c) Using Table III, if f is approaching t, then Δf is approaching Δt, then what is $\frac{\Delta f}{\Delta x}$ approaching?

d) The ratio of $\frac{\Delta f}{\Delta x}$ is approaching $\frac{\Delta t}{\Delta x}$ which is approaching 9. What does this mean?

A differential is the ratio or relationship; the quantities $\frac{\Delta f}{\Delta x}$ approaches the same ratio as $\frac{\Delta t}{\Delta x}$, which means it becomes the slope of the tangent line. For our example, Δf goes to df (the differential of f) and Δx goes to dx (the differential of x). The expression $df = 9dx$ means that in the limit as $x \to 1$, the difference between $f(x)$ and $f(1) = -4$ will approach 9 times the difference between x and 1. This is a relationship not algebraic quantities.

This means our equation can be written three ways using Calculus notation to represent the tangent line:

$$t(x) = f(a) + f'(a)(x - a)$$

$$t(x) = f(a) + f'(a)dx$$

$$t(x) = f(a) + dy$$

Now that we have studied derivatives and tangent lines, we will revisit the average rate of change of volume for a Tootsie Roll® using these methods to find and compare solutions.

Section 7.9 The Chain Rule

Looking Back 7.9

The Power Rule applies to functions that are monomials. Using our newly introduced notation, the Power Rule can be displayed as the following equation:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Power Rule only applies when there is one term that has a constant and an exponent; there is one variable and one exponent, and the exponent is not a variable but a constant.

When y is equal to kx^n and $n > 0$, we say that y is directly proportional to x^n and k is the constant of proportionality or dilation factor.

The Power Rule cannot be applied to the equation $f(x) = 3x - 1$ because it is a binomial. However, the Power Rule can be used to differentiate each of the terms given the Sum Rule…

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

… or the Difference Rule…

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

The Power Rule cannot be applied to the equation $g(x) = 3(4)^x$ because the variable is the exponent. Other rules are used to differentiate more complex functions. We will learn about those in the next few sections of this module. In this section, the Chain Rule will be introduced to differentiate a composition of functions.

But first let us review composite functions. A function is the relationship between two sets of related quantities. The function $y(x)$ is the function machine that turns x into y. So if $y = x^2$ then every value of x corresponds to a unique value y which is its' square. As one machine is hooked to another, we pass the results of one function to another: $x \to [f] \to y$ means that $y = f(x)$ We do this implicitly when we solve $y = 4(x + 3)$. We first add 3 to x and when we get that value, we multiply that value by 4.

Composition of functions is particularly useful when taking the derivative. Let us say we have $x \to [f] \to y \to [g] \to z$ and each separate function is differentiable. Then $f' = \frac{dy}{dx}$ gives the instantaneous rate of change of y with respect to x. Also, the derivative $g' = \frac{dz}{dy}$. Let us say that $f' = \frac{dy}{dx} = 2$ and $g' = \frac{dz}{dy} = 3$. This means that y is growing 2 times as fast as x and z is growing 3 times as fast as y, which means that z must be growing 2·3 or 6 times as fast as x. This can be written:

$$\frac{dz}{dx} = \frac{dz}{dy}\Big|_{y=f(x)} \cdot \frac{dy}{dx}$$

We perform the derivative of z with respect to its input but in the end replace the input by $f(x)$. This becomes a chain of events known as the chain rule.

Let us try the chain rule to find the derivative for $y = (x^2)^2$. The quantities are $x, y,$ and w and the functions are f and g. We will do it the long way to understand the algorithm first and then try the short way.

Let $w = x^2$. Then $y = g(g(x))$, where $w = g(x) = x^2$ or the inner function and $y = g(w) = w^2$ or the outer function:

$$\frac{dy}{dx} = \frac{dy}{dw}\Big|_{w=x^2} \cdot \frac{dw}{dx}$$

$\frac{dy}{dw} =$

$\frac{dw}{dx} =$

And $\frac{dy}{dx} =$ Now replace w by x^2 in the result and simplify.

This confirms our expectations. Now for the short way. Let's fill in the blanks for $y = x^4 = (x^2)^2$.

$$y = (\_\_\_)^2$$

$$\frac{dy}{d\_\_} = \_\_\_\_(\_\_\_\_)$$

$$= 2(\_\_\_\_)\frac{d}{dx}(\_\_\_\_)$$

$$\frac{dy}{dx} = 2x^2 \cdot 2x$$

$$\frac{dy}{dx} =$$

The chain rule has become taking the derivative of the outer function with respect to the inner function and then multiplying by the derivative of the inner function. The Chain Rule states: "For a composition of two functions, $\frac{d}{dx}f(g(x))$ is equal to $f'(g(x))g'(x)$." This simply means we must find the product of the derivative of the outer function multiplied by the derivative of the inner function.

<u>Looking Ahead 7.9</u>

Example 1: Given $f(x) = (\cos x)^2$ find $f'(x)$. (We are finding the first derivative $f'(x) = \frac{df}{dx}$; we are finding the derivative of f with respect to x). Remember that $-2\sin x\cos x = -\sin(2x)$. That is a double angle formula from Trigonometry.

Example 2: Find the derivative of $g(x)$ given $g(x) = (-2x^2 + 5x)^{\frac{1}{3}}$. (This is a composition of functions. Call them $g(h(x))$. We want to find $g'(h(x)) \cdot h'(x)$. Remember, the co-domain is the intersection of two functions. The co-domain of the interior and exterior functions is where they overlap. The derivative is not defined at its endpoints. The domain of g is non-negative real numbers, so the domain of $g(x)$ is $0 \leq x \leq \frac{5}{2}$ and the domain of $g'(x)$ is $0 < x < \frac{5}{2}$.

Example 3: Find the derivative of $h(x)$ given $h(x) = \sqrt[3]{2x+3}$. The domain of h is non-negative real numbers which means the domain of $h(x)$ is $x \geq -\frac{3}{2}$ and the domain of $h'(x)$ is $x > -\frac{3}{2}$.

Section 7.10 The Product Rule

Looking Back 7.10

The Sum and Difference Rule states that "if $f(x)$ is equal to $g(x)+h(x)$, then $f'(x)$ is equal to $g'(x)+h'(x)$."

The Calculus notation for a sum is shown as follows:

$$\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$$

The derivative of the sum of two functions is the same as the sum of the derivatives of the two functions. (Try saying that backwards!)

Do you think this rule applies to the product of two functions? In other words, is the derivative of the product of two functions the same as the product of the derivative of each of the two functions?

If $f(x)$ is equal to $g(x)\cdot h(x)$, is $f'(x)$ equal to $g'(x)\cdot h'(x)$?

Let $g(x)$ be equal to x^3 and $h(x)$ be equal to x^7; then $f(x)$ is equal to $x^3 \cdot x^7$, which is equal to x^{10}, and $f'(x)$ is equal to $10x^9$.

If $g(x)$ is equal to x^3, then $g'(x)$ is equal to $3x^2$.

If $h(x)$ is equal to x^7, then $h'(x)$ is equal $7x^6$.

That makes $g'(x)h'(x)$ equal to $(3x^2)(7x^6)$. This way, $f'(x)$ is equal to $21x^8$.

$$10x^9 \neq 21x^8$$

The derivative of the product of the two functions is $10x^9$, but the product of the derivative of each of the two functions is $21x^8$. They are not the same.

We need to find an equation for the derivative function regarding the product of two functions. This is called the Product Rule.

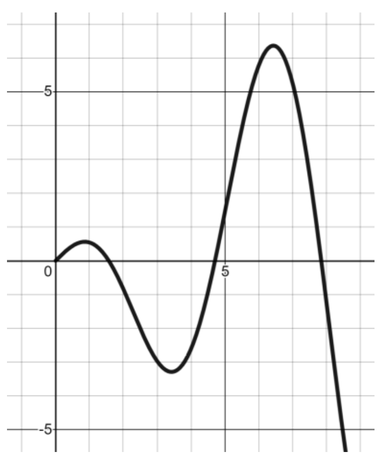
Looking Ahead 7.10

Example 1:	Derive the Product Rule for $f(x)=g(x)h(x)$.

Example 2: For the following function, use the Product Rule on y to find $\frac{dy}{dx}$. Let x^3 be u and x^7 be v.

$$y = x^3 \cdot x^7 = x^{10} \therefore y' = 10x^9$$

Example 3: An object vibrates so that the displacement $d(t)$ is a given number of meters from its resting position by the equation $d(t) = t \cdot \cos t$ in which t represents time in seconds. Use this information and the graph below to solve the problems that follow.



a) How far is the object from its resting position at 7 seconds? At 8 seconds?

Use the “graph trace” feature on the graphing calculator:

When $t = 7$, $d(t) = 5.28$ m. at the point $(7, 5.2)$. At 8 seconds, the distance is approximately -1.16 m. This means the object is moving away from its resting position, not towards it.

b) Find the equation for the velocity of the object, $v(t)$ using the Product Rule.

c) What is the velocity of the object at 7 seconds? At 8 seconds?

By now, you realize that to differentiate means to take the derivative, which is actually finding the slope at any point. Hopefully, you realize the importance of slope as a rate of change as well.

Section 7.11 The Quotient Rule

Looking Back 7.11

We have now learned many rules that help you differentiate functions.

The Constant Rule:

$$\frac{d}{dx}(c) = 0$$

The Power Rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Sum Rule:

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

The Difference Rule:

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

The Chain Rule:

$$\frac{d}{dx}f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$$

The Product Rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Now that we have rules for differentiating sums, differences, and products, it makes sense that we should learn to apply one more rule- the Quotient Rule for differentiating functions.

Looking Ahead 7.11

Example 1: Find the derivative of $\frac{f(x)}{g(x)}$. We are looking for $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$, which can be written "$\frac{d}{dx}(f(x)\cdot(g(x))^{-1})$." Now we can use the Product Rule because we are multiplying values: $\frac{d}{dx}(f(x)\cdot(g(x))^{-1}) = f'(x)(g(x))^{-1} + f(x)\cdot(g'(x))^{-1}$. This may also be written more formally in standard notation as $\frac{df}{dx}\cdot[g(x)]^{-1} + f(x)\cdot\frac{d}{dx}[g(x)]^{-1}$. This is a composite function so we must use the Chain Rule for differentiation.

That is called the Quotient Rule. Notice that the numerator of the Quotient Rule looks very much like the Product Rule, except a subt. The denominator of the Quotient Rule is the original denominator squared. Let us apply this knowledge in the next example!

Example 2: Find $\frac{d}{dx}\ \frac{3x^2}{5x^4}$. Treat $f(x) = 3x^2$ as the numerator functions and $g(x) = 5x^4$ as the denominator.

Example 3: Find the derivative of the trigonometric function $\tan(x) = \frac{\sin(x)}{\cos(x)}$ by recognizing that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Section 7.12 Derivatives of Trigonometric Functions

Looking Back 7.12

Previously, we have learned that $\frac{d}{dx}\sin x$ is equal to $\cos x$ and $\frac{d}{dx}\cos x$ is equal to $-\sin x$. Though we did not prove these derivatives, we demonstrated them graphically by finding the slope of the tangent lines at various values of x. Using that same method, we could demonstrate the following:

$$\frac{d}{dx}(-\sin x) = -\cos x$$

$$\frac{d}{dx}(-\cos x) = \sin x$$

Now, we are back to where we started from. It may be easier to memorize these derivatives and then solve them graphically each time. In Example 3 of Section 7.11, we used the Quotient Rule to differentiate the following function:

$$\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{1}{\cos^2 x}$$

Because $\tan x$ is equal to $\frac{\sin x}{\cos x}$, then $\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$. Because $\frac{1}{\cos^2 x}$ is equal to $\sec^2 x$, then $\frac{d}{dx}\tan x = \sec^2 x$.

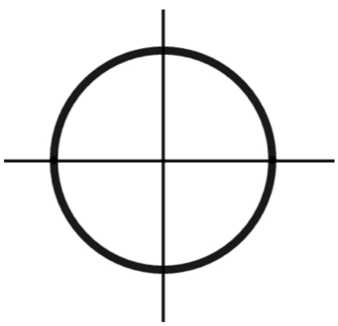
Other patterns emerge as we find the derivatives of the remainder of the six trigonometric functions: sin, cos, tan, csc, sec, and cot.

We will be finding $\frac{d}{dx}(\tan x)$ and $\frac{d}{dx}(\cot x)$ in the Practice Problems section. We know that $\cot x$ is equal to $\frac{1}{\tan x}$. Remember that $\sec x$ is equal to $\frac{1}{\cos x}$, $\csc x$ is equal to $\frac{1}{\sin x}$, and $\cot x$ is equal to $\frac{\cos x}{\sin x}$; this will be helpful when we find the derivatives of the other trigonometric functions.

Looking Ahead 7.12

Example 1: Find $\frac{d}{dx}\csc x$. We know that $\csc x$ is equal to $\frac{1}{\sin x}$. We can use the Quotient Rule or chain rule to find $\frac{d}{dx}(\frac{1}{\sin x})$.

Example 2: Find the derivative of $x^2 + y^2 = 1$. We know that the graph of this equation is the unit circle because the unit circle is the graph of all points that satisfy this equation.



The slope of the tangent lines at every point around the unit circle would give us the graph of the derivative. This equation/graph is not a function because for every input value of x, there are two output values for y. As we learned in Algebra 2, the function $x^2 + y^2 = 1$ can be written "$y = \sqrt{1 - x^2}$" or "$y = -\sqrt{1 - x^2}$."

Then the derivatives can be taken separately using explicit differentiation. On the other hand, the derivatives can be found using implicit differentiation with the Chain Rule. The first step is to differentiate both sides of the equation. We will need to know four things at the outset.

$$\frac{d}{dy}\, y^2 = \frac{dy^2}{dy} = 2y$$

This is because of the Power Rule and represents the derivative of y^2 with respect to y.

$$\frac{d}{dx}\, x^2 = \frac{dx^2}{dx} = 2x$$

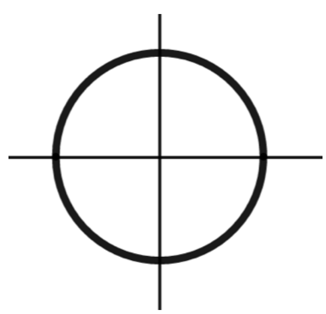
This is because of the Power Rule and represents the derivative of x^2 with respect to x.

$$\frac{d}{dx}\, 1 = \frac{d(1)}{dx} = 0$$

This is because of the Constant Rule and represents the derivative of 1 (which can be written "cx^0" with respect to x in which c is a constant.)

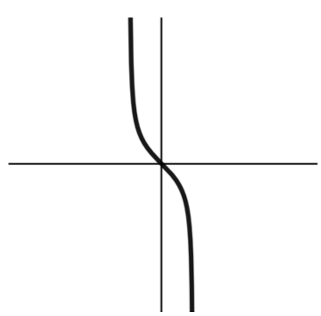
$$\frac{d}{dx}y = \frac{dy}{dx}$$

We do not know what $\frac{dy}{dx}$ is; it is what we are trying to find. By finding the derivative, $\frac{dy}{dx}$, we are finding the slope for any point on the unit circle.



The point on the derivative curve is $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ for a 45° angle on the unit circle.

The graph of the derivative of $x^2 + y^2 = 1$ is quite intriguing.



Our God is a very intriguing God. He is a God of design and order. His patterns modeled in the universe are intricate and detailed. How wonderful that He gives us many ways to solve problems- it is just another way He displays His splendor!

Let us use implicit differentiation to find the derivative of an inverse trigonometric function.

Example 3:	Find the derivative of the inverse function $y = \cos^{-1}x$.

Now the derivative is in terms of y, but we need it in terms of x. The trigonometric identity $\sin^2 y + \cos^2 y = 1$ will help us find the derivative in terms of x.

(When we take the square root of both sides)

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y}$$

This is useful to us because we know that $\cos y$ is equal to x so $\cos^2 y$ is equal to x^2. Therefore, we can substitute those values under the radicand and the equation becomes $\sin y = \sqrt{1 - x^2}$.

Now, we can go back to $\frac{dy}{dx} = -\frac{1}{\sin y}$ and substitute $\sqrt{1 - x^2}$ for $\sin y$, which results in the following equation:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$$

Now we have our derivative in terms of x. We could graph this as well. Because we know that $\frac{d}{dx}(\cos x) = -\sin x$ and $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}(\sin x) = \cos x$, we can look at the patterns and understand the derivative of $\sin^{-1}x$; it is $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$.

This is yet another display of God's design in the universe!

Section 7.13 Derivatives of Exponentials and Logarithms

Looking Back 7.13

In the last section, we saw some remarkable patterns emerge for the derivatives of the trigonometric functions and used them to prove some of the derivatives for the inverse trigonometric functions. They are listed for you below.

Derivatives of the Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

Derivatives of the Inverse of Trigonometric Functions:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

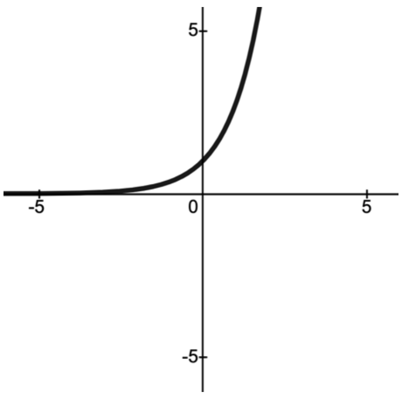
Now that they are listed, the patterns are easily identified. This is all evidence of orderly and intelligent design in our universe established by an orderly and intelligent Designer.

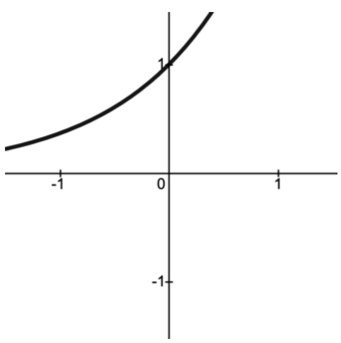
In this section, the final section of the module, we will explore the derivatives of exponentials and their inverses, logarithms.

We will investigate e and its derivative. We learned that e is a transcendental number much like π and φ (phi). These numbers are found throughout the natural world and often used in real-world problem-solving. They are actually quite remarkable, and to me, they are powerful examples of an omnipotent Designer.

Looking Ahead 7.13

Example 1:	Find the derivative of e^x graphically.



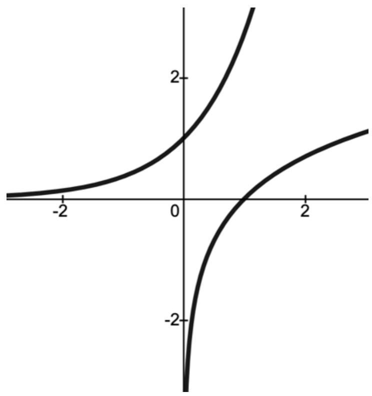


On the graph of e^x, when x is equal to 1, $e^1 = e$; e is just a constant (2.71828 ...) and a constant to the first power is just a constant. Amazingly, when x is equal to 1, the slope of the tangent line at 1 is also $2.71825\ldots$ If we continued the table, we would find that the values of the slope of the tangent line at x are equal to e^x at each value of x. Isn't that remarkable?! In all of mathematics, that is astounding. Just as $y = x$ is its own inverse and the line of reflection for all other functions, e^x is its own derivative!

$$\frac{d}{dx}(e^x) = e^x$$

Just stop for a minute and ponder that; give an awe-inspired moment to the wonderful, intelligent Designer of the universe- the true author and Master of Mathematics!

It does not end there. Another amazing derivative is the derivative of the inverse natural logarithm function, $\ln x$.



Therefore, $\ln x$ is reflected over the line $y = x$, which is its own inverse! Are you making the connections to how vast, yet logical mathematics is!

When $e^y = x$, $\ln(x) = \log_e(x) = y$ and $e \approx 2.718$

The natural logarithm function $\ln(x)$ is the inverse function of the natural exponential function e^x when $x > 0$.

$f(f^{-1}(x)) = e^{\ln(x)} = x$ or $f^{-1}(f(x)) = \ln(e^x) = x$ because $\log_e(e^x) = y$ and $e^y = e^x$ so $y = x$

The derivative of an exponential function is:

$$\frac{d}{dx}(a^x) = (a^x)\ln a$$

Using the Chain Rule, the derivative of a constant to the power of x is the constant to the power of x multiplied by the natural logarithm of the base of the exponential, which is a constant. That is true for e^x as well:

$$\frac{d}{dx}(e^x) = (e^x)\ln e$$

We know that $\ln e^1$ is equal to 1. The inverse of the natural logarithm to the first power is just 1. Because $\frac{d}{dx}(e^x) = e^x \cdot (1)$, we write it $\frac{d}{dx}(e^x) = e^x$.

Example 2: Differentiate 3^x. We want to find $\frac{d}{dx}3^x$ in which a is 3. Our base is the constant 3.

Example 3: Differentiate $h(x) = 3e^{\cos(x)\sin(4x)}$. This will require us to use the product rule and the chain rule, in addition to what we have learned of the derivatives of trigonometric and exponential functions.