Pre-Calculus and Calculus Module 4 Polar Equations and Complex Numbers

Section 4.1 More on Parametric Equations

Looking Back 4.1

In the last module, we began with learning about vectors and ended with a study of parametric equations.

Previously, we have studied equations by exploring y in terms of x and have also graphed them. We learned that x can be written in terms of y. The former equations (prior to our study of vectors) usually used these two variables on the x-y coordinate plane. We also investigated x, y, and z on the 3-dimensional x-y-z plane.

A parametric equation adds a third (or fourth) variable to the mix. This variable is a helping parameter, often t, which represents time.

Parametric equations are particularly helpful when exploring projectile motion, as seen in firing a cannonball off a cliff. The path is a parabola, but there are two combined motions the cannonball follows: the horizontal motion is x and the vertical motion is y.

These can be represented parametrically by two functions:

$$x = g(t)$$
$$y = h(t)$$

In these two functions, x is horizontal travel over a defined interval and y is the height of the cannonball.

A parametric equation models this path. The variable t controls both variables x and y so t is independent and x and y are both dependent on t. However, t is not used to plot the graph, only x and y are used to plot the graph.

In this module, we will see how parametric equations can be used with polar equations as well (once we learn what polar equations are).

First, let us continue our discussion of parametric equations.

Looking Ahead 4.1

In the last module, you converted parametric equations to rectangular equations. Here are some things to consider when doing this: to eliminate the parameter t, you must first solve for t in one equation and then substitute that in the other equation and solve for y.

Example 1: Write the parametric equations as one rectangular equation:

$$x = 2t + 1$$
$$y = t + 3$$

Example 2: Use the same equations from Example 1 but solve for t in the first equation ($x = 2t + 1$) and substitute that for y in the second equation ($y = t + 3$). Do you get the same equation as in Example 1?

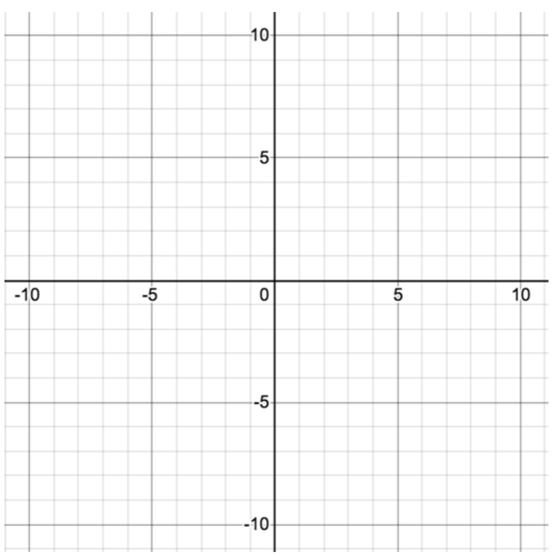
Example 3: Graph the parametric equations over the interval $0 \leq t \leq 4$:

$$x = 2t + 1$$
$$y = t + 3$$

Now, graph the equation $y = \frac{1}{2}x + \frac{5}{2}$ using the values of x over the interval $0 \leq t \leq 4$ to solve for y. What do you notice?

t	x	y	(x, y)
0			
1			
2			
3			
4			

x	y	(x, y)



Section 4.2 The Helping Parameter

Looking Back 4.2

The parameter t in a parametric equation can help us understand vertical and horizontal motion of a projectile or the horizontal distance and height of the projectile at any given time t.

Although t is often the helping parameter, it is not always the third variable. Sometimes, the helping parameter may be r (radius). We will see examples of this when we look at some unique polar graphs.

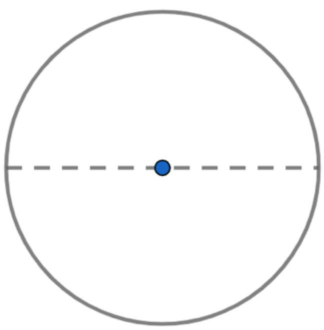
For now, let us take our first steps in getting there by doing what we always do- building upon our prior knowledge.

Looking Back 4.2

You are building a dollhouse pool that is round and has a diameter of 8 inches. You start painting at point $(4, 0)$ on the edge of the pool. If the x-coordinate of your paintbrush is $x = 4\cos t$, what is the y-coordinate of your paintbrush after t minutes? The y-coordinate would be $y = 4\sin t$. You start painting at $t = 0$ and paint 4 inches in 1 minute.

(You may be remembering all of this from your work with the unit circle in Geometry and Trigonometry, or your review in the previous practice problems section.)

Example 1: Using the example in the Looking Back section, why is the angle from your start point (the center of the pool) to the endpoint of your paintbrush 1 radian after 1 minute?

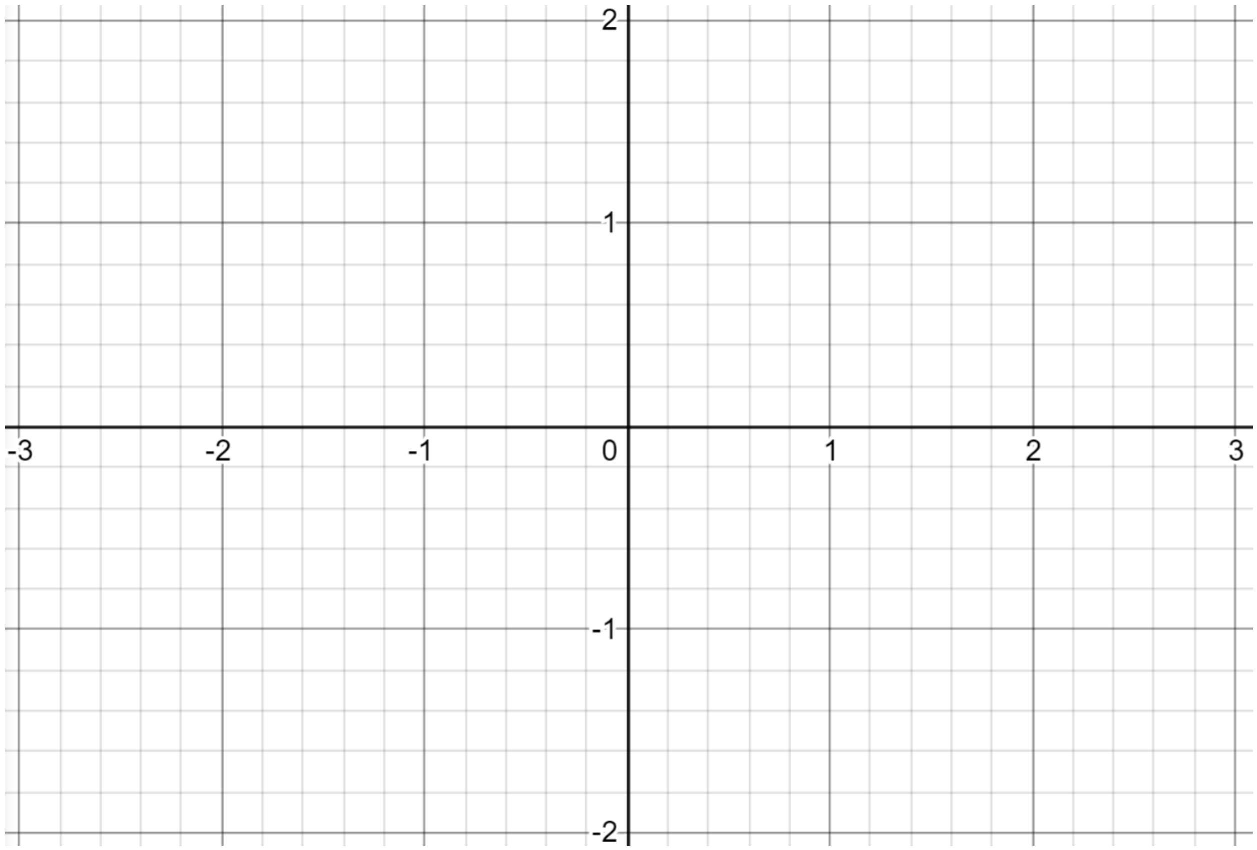


How many radians would you paint in 2 minutes? How many inches would that be?

Example 2: Use your calculator to complete the given table. Go to "Lists and Spreadsheets" and let Column A be t. Put in the values from the table. Let Column B be x and type in the formula $x = t\cos t$ for Column B. Let Column C be y and type in the formula $y = t\sin t$ for Column C. Round the coordinates to the nearest hundredths.

$\boldsymbol{t}$	$\boldsymbol{x}$ $\boldsymbol{t\cos t}$	$\boldsymbol{y}$ $\boldsymbol{t\sin t}$	$(\boldsymbol{x}, \boldsymbol{y})$
0			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			
$\frac{\pi}{2}$			
$\frac{2\pi}{3}$			
$\frac{3\pi}{4}$			
π			

Example 3:	Plot the points (x, y) from Example 2 on the given graph. What shape do you get?



It is a curved line, but this is only from $t = 0$ to π . This could be continued over the interval $\pi \leq t \leq 4\pi$.

Go to the parametric mode on the graphs page of the calculator to see the full graph. To do this, go to…

- "Graph"
- "Menu"
- "Graph Type"
- "Parametric"

Then, type in…

- $x_1(t) = t \cdot \cos t$
- $y_1(t) = t \cdot \sin t$
- $0 \leq t \leq 4\pi \; t\text{step} = 0.13$

Note: Make sure to type "·" between the t and $\cos t$, and t and $\sin t$.

Hit "Enter."

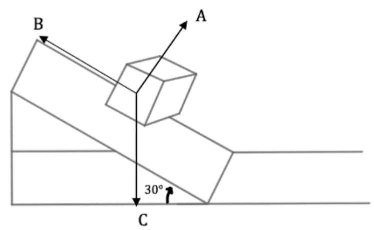
Now, that is an interesting design! It looks like the shell of a snail. And we again see the beautiful mathematics in God's natural world.

This is an Archimedean or Arithmetic Spiral.

Section 4.3 Application Problems

Looking Back 4.3

A loading ramp has an angle of $30°$ with the ground. A 100 kg. box slides down a given ramp with constant velocity. This means the sum of the vectors in A, B, and C is 0.



If we want to find the frictional force acting on the box, we must first find the gravitational force acting on the box (vector C), which is…

$$\|\mathrm{C}\| = 9.8\ \mathrm{N/kg.} \cdot 100\ \mathrm{kg.} = 980\ \mathrm{N}\ \text{(where N is newtons)}$$

Using the properties of trigonometry, $\sin 30° = \frac{\|\mathrm{B}\|}{\|\mathrm{C}\|}$.

Solving for $\|\mathrm{B}\|$, the frictional force is…

$$\|\mathrm{C}\| \sin 30° = \|\mathrm{B}\|$$

$$980\ \sin 30° = \|\mathrm{B}\|$$

$$490 = \|\mathrm{B}\|$$

The frictional force on the box is 490 N (newtons) up the ramp.

Looking Ahead 4.3

Just as you used properties of trigonometry to solve work and force problems, you can also use properties of vectors to solve work and force problems.

Example 1: Using the formula $\mathrm{w} = \mathrm{F} \cdot d$ (this equation assumes constant force), find the work done to move a box from point H to J if the object moves in meters from $\mathrm{H}(1, 3)$ to $\mathrm{J}(6, 4)$, and the force vector (in newtons) is $\mathrm{F} = 5\boldsymbol{i} - 2\boldsymbol{j}$.

Joules (J) is the term used for work, which is $\text{Newton} \cdot \text{meter}$. This was named in honor of James Prescott Joule, a gifted physicist and highly esteemed Christian. Trained by English chemist (and also devoted Christian) John Dalton, Joule was a creationist scientist. He believed the study and pursuit of God were first and foremost in life, followed by the study of God's handiwork. Joule believed that in knowing the natural laws of the universe, we know something of the mind of God.

Nowadays, Joule is recognized as the chief founder of thermodynamics, though others- like William Thomson, a professor at the University of Glasgow- contributed greatly to the field (he was another devoted Christian). It was Joule's principle of conservation that formed the basis for thermodynamics. The first law of thermodynamics states that energy cannot be created or destroyed, simply changed from one form to another. Joule knew that which God created could not be destroyed by man.

Knowing that, let us look at an example involving energy.

Example 2: A cable is used to hoist a crate directly up 60 meters to the restaurant of a hotel. How much energy (in kilowatt-hours) is needed to raise the 250 kg. crate? Watt is a unit of power-rate of energy transfer of 1 joule per second. 1Kwh=3600 KJ (1 kilowatt-hour= 3600 joules).

Section 4.4 The Polar Coordinate System

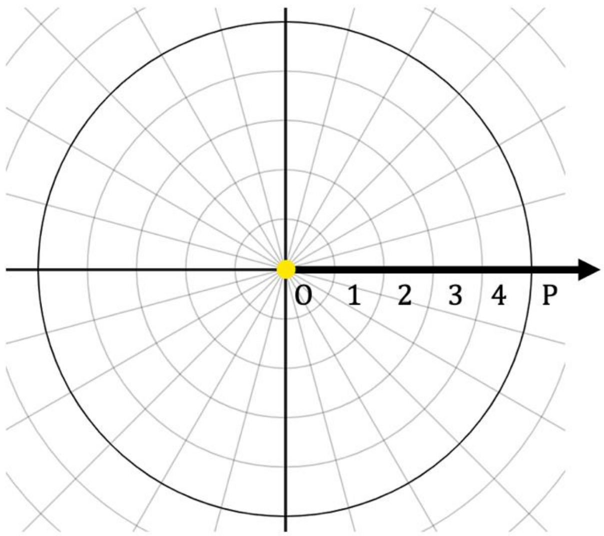
Looking Back 4.4

At the end of the second section, you graphed an interesting design that looked like the shell of a snail. In this section, we are going to introduce polar equations. You will see many more interesting designs that can be made using polar equations as we continue in this module.

It is fascinating how many mathematical concepts are related and how they build on one another. In the last module, we learned how vectors are related to parametric equations. In this module, we will learn how parametric equations are related to polar graphs.

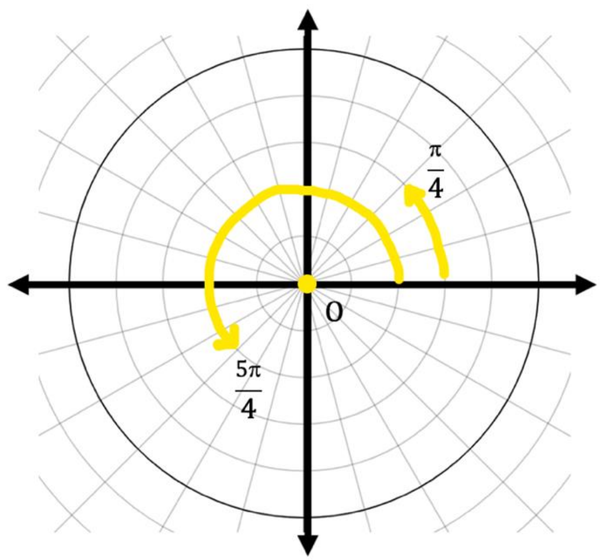
Looking Ahead 4.4

The x-y coordinate plane has a horizontal axis of x and a vertical axis of y. The component form of vectors is represented by the variables $\boldsymbol{i}$ and $\boldsymbol{j}$. Now we are ready to introduce polar graphs, which use the radius of a circle (r) and the angle of rotation (in degrees) to locate a point on the graph. A polar graph looks like this:

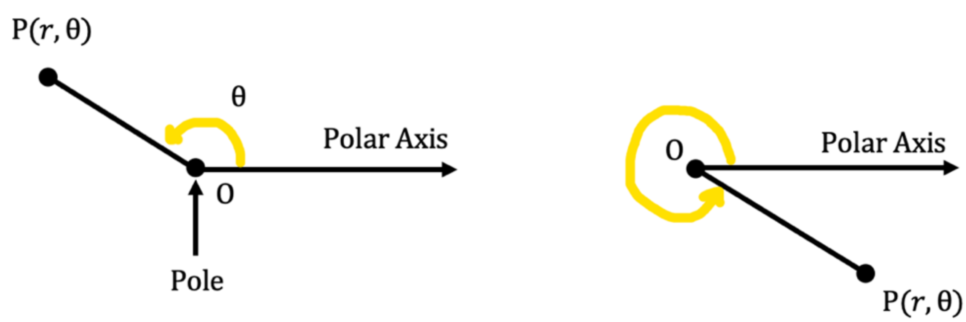


The polar coordinate system has a point O at the center, which is called the pole (it is highlighted in the graph above). Rather than horizontal and vertical grids, there are arcs and circles of one unit, two units, three units, etc. along the ray. These move horizontally along the pole. The arrow in the graph above represents the polar axis. This is the name for the x-axis when working with polar graphs.

The pole is the reference point of a polar graph, and the polar axis is the ray from the pole in the reference direction. The polar angle moves in a counterclockwise direction and may be in radians or degrees.

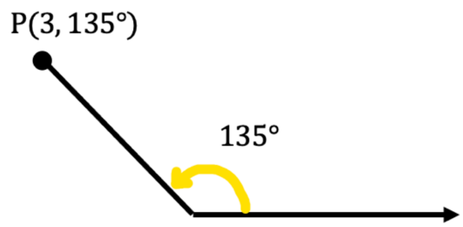


The polar coordinate system also has a polar axis with O as its endpoint. The polar coordinates are the ordered pair (r, θ), where $r = \mathrm{OP}$ and θ is the measure of the angle from the polar axis to the segment OP, which can be measured along the rings that form the circumference of each circle. This line segment OP is the radius from the origin (pole) to point P.

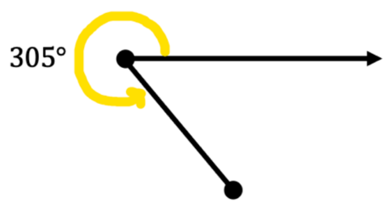


To find the terminal side of the polar angle, first locate the angle from the polar axis. Plot the point r units (rings) in the direction of the angle from the pole. If r is negative, plot the point $|r|$ units in the opposite direction.

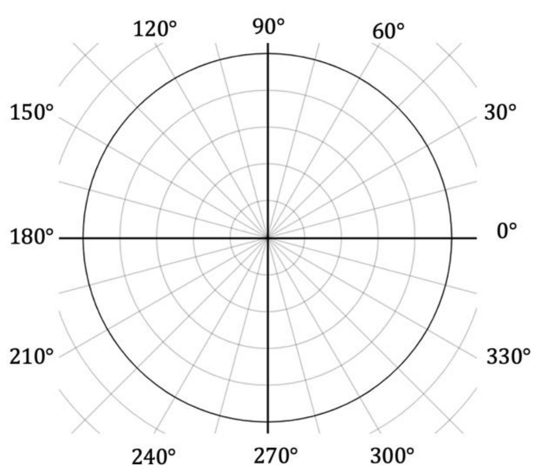
If $r = 3$ and $\theta = 135°$, then point P is $(3, 135°)$.



If $r = 4$ and $\theta = 305°$, then point P is $(4, 305°)$.

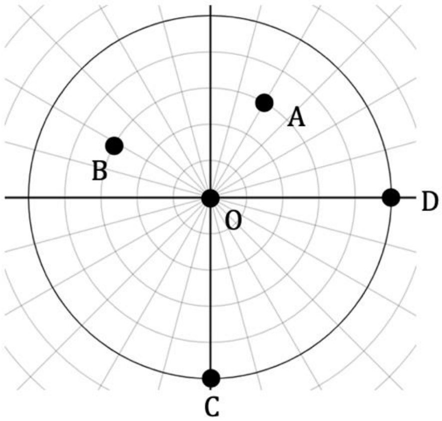


The angles on the polar coordinate system are those of the unit circle.

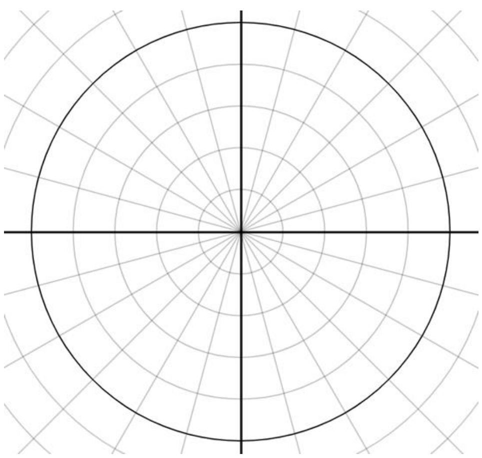


The first coordinate, r, can be thought of as the radius of the circle with O as the center. This is measured in units.

Example 1:	Name the ordered pairs for A, B, C, D located on the polar coordinate grid below.



Example 2:	On the polar coordinate grid below, locate and label the points given.



$E(4.5, 90°)$

$F(3, 105°)$

$G(-3, 90°)$

$H(0, 90°)$

Section 4.5 Plotting Polar Coordinates

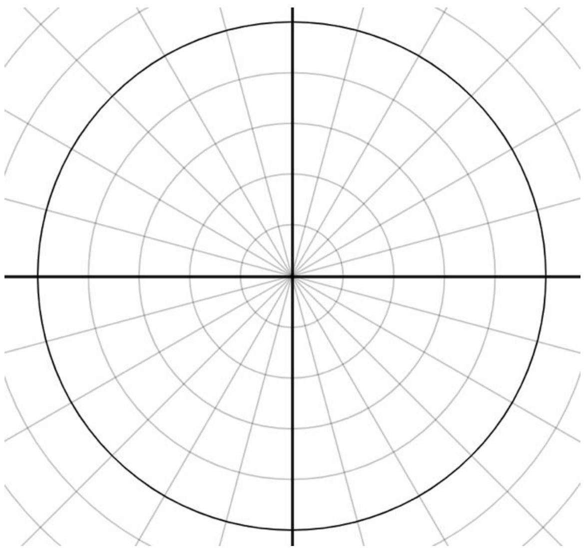
Looking Back 4.5

In the previous Practice Problems section, you learned that the point $(5, 270°)$ could also be named $(-5, 90°)$. Whether r is positive or negative, the angle is counterclockwise from the polar axis when θ is positive. If θ is negative, then the angle is clockwise from the polar axis.

Therefore, there is more than one way to name a point.

Looking Ahead 4.5

Example 1: Name the point $(1, 30°)$ three other ways. Let r represent the distance from the pole and θ represent the angle formed from the polar axis.



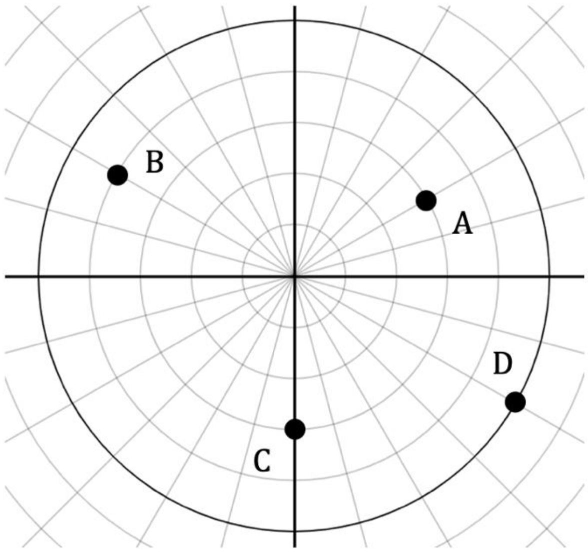
Notice that a $30°$ angle counterclockwise from the polar axis is also $330°$ clockwise from the polar axis: $30° - 360° = -330°$. So, when you find the angle, plot the point r units in the direction of the angle. However, if r is negative, move $|r|$ units in the opposite direction of the angle to plot the point.

Notice also that the angle opposite θ is $\theta \pm 180$. To find $-r$, move to the $|r|$ in the opposite direction.

Therefore, the four forms of the polar coordinates are shown as follows:

$$(r, \theta)$$
$$(r, \theta - 360°)$$
$$(-r, \theta - 180°)$$
$$(-r, \theta + 180°)$$

Example 2: Label the polar coordinate grid below in radians rather than degrees. Name the four points on the polar coordinate grid using (r, θ), where θ is in radians. Then name each of the points three other ways.



In the previous section, you picked angle measurements for θ and then determined the corresponding r values to graph a polar equation when you plotted the points from the table. Finding the symmetry of an equation allows you to use fewer points to plot the graph. Just as there are three tests for symmetry with rectangular coordinates, there are three tests for symmetry for polar coordinates:

1. If an equivalent equation results when you replace θ with $-\theta$, then the graph is symmetric with respect to the polar axis.
2. If an equivalent equation results when you replace θ with $-\theta$ and r with $-r$, then the graph is symmetric with respect to $\theta = \frac{\pi}{2}$.
3. If an equivalent equation results when you replace r with $-r$, then the graph is symmetric with respect to the pole.

In other words:

1. Reflection symmetry about the horizontal axis (x-axis).
2. Reflection symmetry about the vertical axis (y-axis).
3. Point symmetry about the pole.

Example 3: Name the symmetry for the equation $r = 3\sin\theta$.

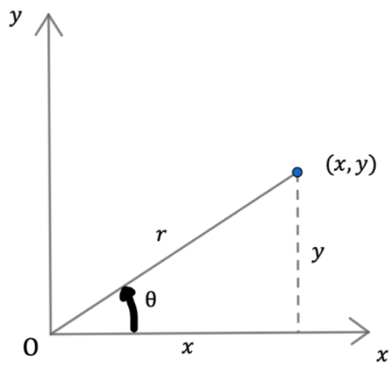
Example 4: Name the symmetry for the equation $r = 1 - 4\cos\theta$.

Note: A polar equation can fail the test for symmetry, but still exhibit the symmetry when it is graphed over a full period of the trigonometric function.

Section 4.6 Converting from Rectangular to Polar Coordinates

Looking Back 4.6

The rectangular coordinate (x, y) is related to the polar coordinate (r, θ) as seen in the diagram below:



The x-coordinate is the adjacent side of the right triangle:

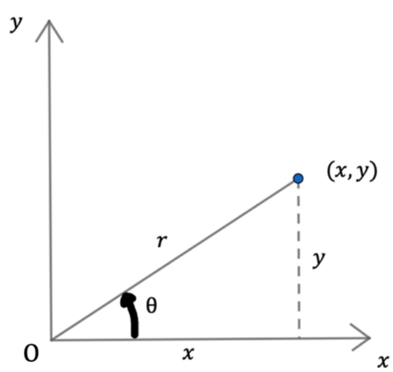
$$\cos\theta = \frac{x}{r}$$
$$r\cos\theta = x$$

The y-coordinate is the opposite side of the right triangle:

$$\sin\theta = \frac{y}{r}$$
$$r\sin\theta = y$$

Use these formulas to convert from polar coordinates to rectangular coordinates.

To convert from rectangular coordinates to polar coordinates, use the Pythagorean Theorem to derive a formula.



$$x^2 + y^2 = r^2 \qquad \cos\theta = \tfrac{x}{r} \Longrightarrow r\cos\theta = x$$

$$\pm\sqrt{x^2 + y^2} = r \qquad \sin\theta = \tfrac{y}{r} \Longrightarrow r\sin\theta = y$$

Looking Ahead 4.6

Example 1:	Convert $S(-2, 60°)$ to rectangular coordinates.

Example 2:	Convert $T(5, -5)$ to polar coordinates.

Section 4.7 Converting Between Polar and Rectangular Equations

Looking Back 4.7

There were a few formulas in the last section that were helpful in converting between rectangular and polar coordinates: $x = r\cos\theta$; $y = r\sin\theta$; $r^2 = x^2 + y^2$.

A relationship that is helpful when converting between rectangular and polar coordinates is: $\tan\theta = \frac{y}{x}$. This means…

$$\tan\theta = \frac{r\sin\theta}{r\cos\theta} \qquad \tan\theta = \frac{\cancel{r}^{1}\sin\theta}{\cancel{r}_{1}\cos\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

(This is a relationship you have previously learned.)

Let us use these same principles to convert between equations in the examples.

Looking Ahead 4.7

Example 1: Convert the rectangular equation $x^2 + y^2 = x + 2y$ to a polar equation. Substitute in polar values for those given.

Example 2: Convert the polar equation $\frac{\sin^2\theta}{\cos\theta} = r$ to a rectangular equation.

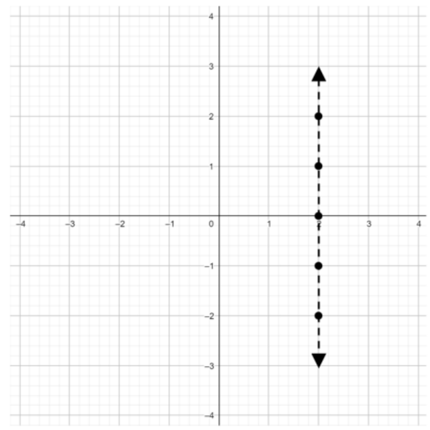
Example 3: Convert the polar-coordinate equation $r(1 + \sin\theta) = 2$ to a rectangular-coordinate equation.

Section 4.8 Graphs of Polar Equations

Looking Back 4.8

We know that $x = 2$ is a vertical line because no matter what the output (y-value) is, the input (x-value) is always 2. The table and graph for $x = 2$ are shown below:

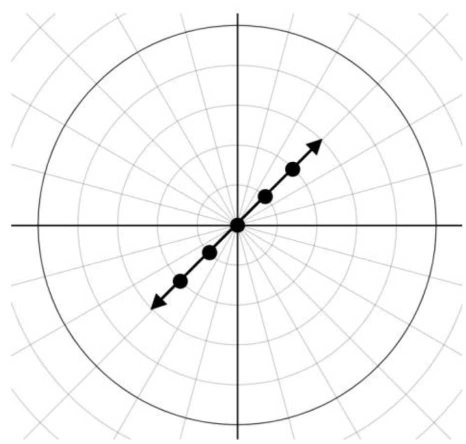
x	y
2	-2
2	-1
2	0
2	1
2	2



Using the same reasoning, the graph of $y = 2$ is a horizontal line.

What do you think the graph of $\theta = \frac{\pi}{4}$ looks like? The table and graph of $\theta = \frac{\pi}{4}$ are shown below:

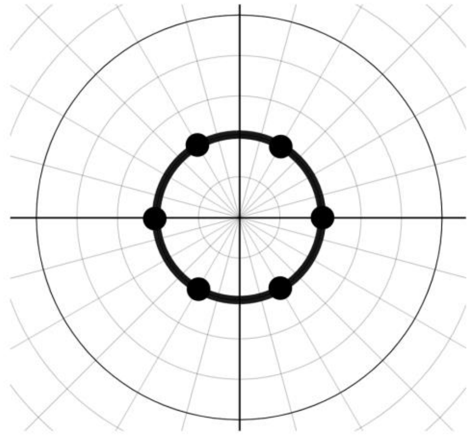
r	θ
-2	$\frac{\pi}{4}$
-1	$\frac{\pi}{4}$
0	$\frac{\pi}{4}$
1	$\frac{\pi}{4}$
2	$\frac{\pi}{4}$



The equation $\theta = \frac{\pi}{4}$ is a diagonal line along the line $\frac{\pi}{4}$ from the polar axis. There are many similarities between rectangular-coordinate point equations and graphs, and polar-coordinate point equations and graphs.

What do you think the graph of $r = 2$ looks like? In this case, as θ changes r stays the same, so it would not be a straight line but a curved line around the circumference of the circle at $r = 2$.

r	θ
2	$\frac{\pi}{3}$
2	$\frac{2\pi}{3}$
2	π
2	$\frac{5\pi}{3}$
2	$\frac{7\pi}{3}$
2	2π



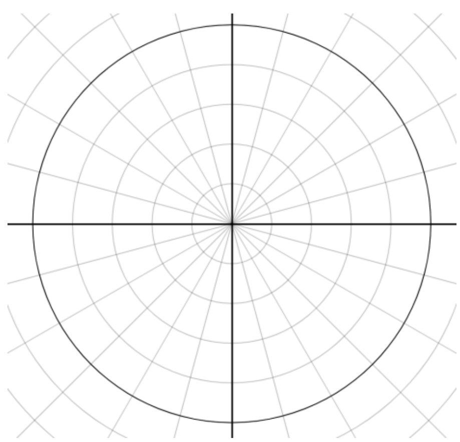
Looking Ahead 4.8

Example 1: Graph $r = 3\theta$ on the polar-coordinate graph. To do this, first make a table. Use radians or degrees.

θ	r is 3θ
$\frac{\pi}{6}$	$\frac{\pi}{2}$
$\frac{\pi}{3}$	π
$\frac{\pi}{2}$	$\frac{3\pi}{2}$
$\frac{2\pi}{3}$	2π
$\frac{5\pi}{6}$	$\frac{5\pi}{2}$
π	3π

θ	r
$\frac{\pi}{6}$	1.57
$\frac{\pi}{3}$	3.14
$\frac{\pi}{2}$	4.71
$\frac{2\pi}{3}$	6.28
$\frac{5\pi}{6}$	7.85
π	9.42

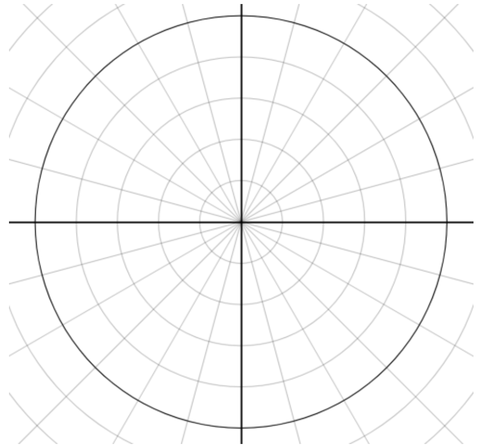
θ	r
30°	1.57
60°	3.14
90°	4.71
120°	6.28
150°	7.85
180°	9.42



When $r = 0$ then $\theta = 0$, then the polar point is (0,0).

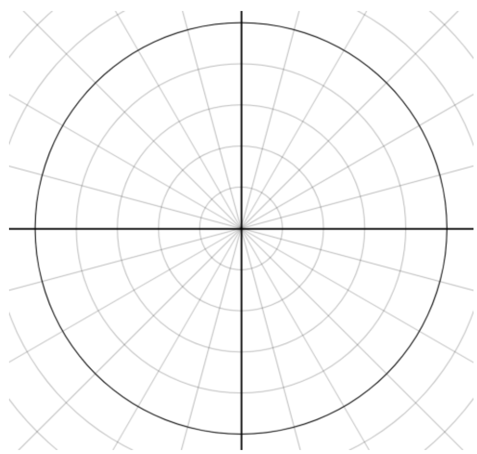
Example 2: Use degrees to complete the table and graph below for the equation $r = \sin(3\theta)$.

$\boldsymbol{\theta}$	$\mathbf{3\theta}$	$\mathbf{sin(3\theta)}$
30°		
60°		
90°		
120°		
150°		
180°		



Example 3: Graph $r = 3\sin 3\theta$. Try to guess what the design will look like before completing the table and design.

$\boldsymbol{r}$	$\mathbf{3\sin 3\theta}$
30°	
60°	
90°	
120°	
150°	
180°	



Just as rectangular equations have shapes that define them, polar equations have shapes that define them. Below are a few that we have investigated so far.

Circles

1. If $r = a\cos\theta$, then a is the diameter of the circle with the leftmost edge centered at the pole.
2. If $r = a\sin\theta$, then a is the diameter of the circle with the bottommost edge centered at the pole.

For cosine, the $-a$ reflects the graph over $\theta = \frac{\pi}{2}$.
For sine, the $-a$ reflects the graph over the polar axis.

Rose Curves

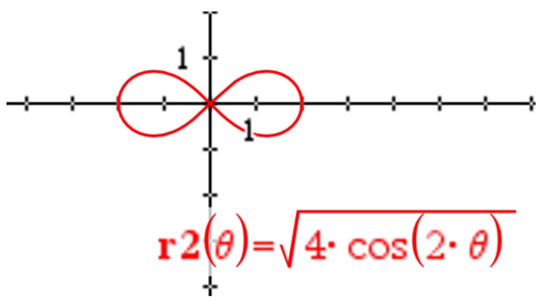
1. If $r = a\cos n\theta$, then $a \neq 0$ and n is an integer greater than 1.
2. If $r = a\sin n\theta$, then $a \neq 0$ and n is an integer greater than 1.

If n is an even integer, then the rose has $2n$ petals.
If n is an odd integer, then the rose has n petals.

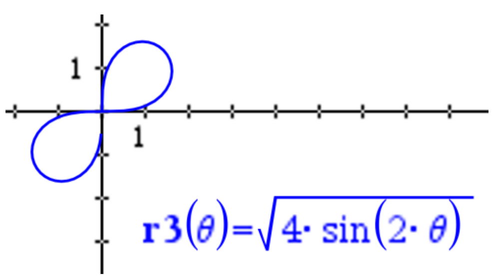
We will now investigate two more polar equations using the graphing calculator.

Lemniscates
Lemniscates have the shape of a propeller or figure-eight.

1. If $r^2 = a^2\cos 2\theta$ in which $a \neq 0$, then this is symmetric to the polar axis, to $\theta = \frac{\pi}{2}$, and to the pole. It has the shape of a figure-eight.



2. If $r^2 = a^2\sin 2\theta$ in which $a \neq 0$, then this is symmetric to the pole. It has the shape of a propeller.



<u>Conic Sections</u>

The shape of the graph will depend on the eccentricity (e).

1. If $0 \leq e \leq 1$, then the graph is an ellipse.
2. If $e = 1$, then the graph is a parabola.
3. If $e > 1$, then the graph is a hyperbola.

If the focus of the conic section is at the origin, and the directrix is $x = \pm p$ in which p and e are positive real numbers, then…

$$r = \frac{e \cdot p}{1 \pm e\cos\theta}$$

If, under the same conditions, the directrix is $y = \pm p$, then…

$$r = \frac{e \cdot p}{1 \pm e\sin\theta}$$

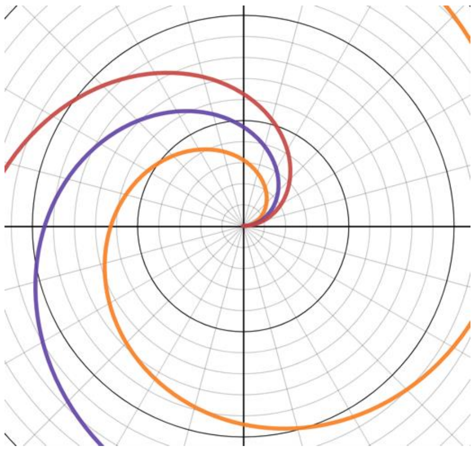
In summary, here are some steps that may be used to graph polar equations:

1. Check the equation for three types of symmetry: the polar axis, $\theta = \frac{\pi}{2}$, and the pole.
2. Set r equal to 0 and find the zeroes.
3. Find the maximum value by using the parent trigonometric function.
4. Make a table of values for r and θ.
5. Plot the points and stop when the symmetry of the equation may be used to find the rest.
6. Sketch the graph by connecting the points.

Section 4.9 Shifts of Polar Graphs

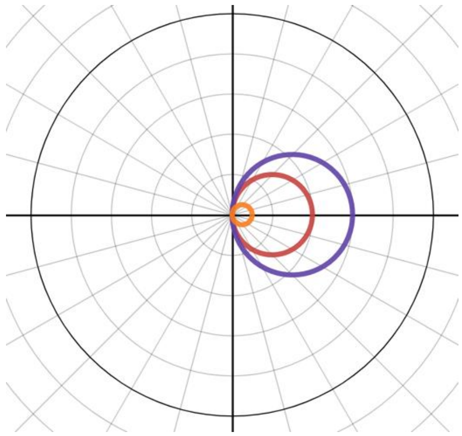
Looking Back 4.9

The graphs of $r = 2\theta$, $r = 3\theta$, and $r = 4\theta$ are color coded for you on the polar-coordinate graph below. How are they similar and how are they different?



The graphs are all spirals where $\theta > 0$. As the coefficient of θ increases, the wideness of the spiral increases. The smaller the coefficient is, the more tightly knit the spiral is. These are called Archimedes Spirals and increase at a constant rate in an ever-widening, never ending spiral.

The graph of $r = 2\cos\theta$ is a circle with a diameter of 2 units. The graph of $r = 3\cos\theta$ has a diameter of 3 units and a larger area than the previous equation. The number being multiplied by $\cos\theta$ dilates (shrinks) the circle depending on the largeness or smallness of the number. What can you say about $r = 0.5\cos\theta$?



The equation $r = 0.5\cos\theta$ is a circle with a diameter of 0.5.

Looking Ahead 4.9

Example 1: Given the graph of $r = f(\theta)$ in which f is a function and $r = f(\theta - \frac{\pi}{2})$ rotates the graph 90° coutnerclockwise, what would $r = f(\theta + \frac{\pi}{2})$ do to the graph?

Slides and translations are fairly easy to do on the rectangular-coordinate grid, but rotations are fairly difficult. On the other hand, slides and translations are fairly difficult to do on the polar grid, but rotations are fairly easy. It is for this reason the polar grid is used for measuring the earth's rotation or mapping the stars.

One application of polar coordinates is to represent a planet's instantaneous position at a given point in time where r is the distance from the sun.

Example 2: Given the graph of $r = 4\sin\theta$, what would the graph of $r = 4\sin(\theta + \frac{\pi}{4})$ look like?

Section 4.10 Graphing Polar Functions Using Technology

Looking Back 4.10

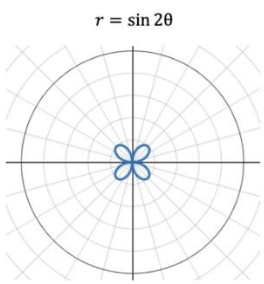
We have previously learned about the polar coordinate $(2, \frac{\pi}{2})$ in which $r = 2$ (r represents the distance from the pole), and θ is equal to $\frac{\pi}{2}$ (which represents the counterclockwise angle of rotation from the horizontal polar axis).

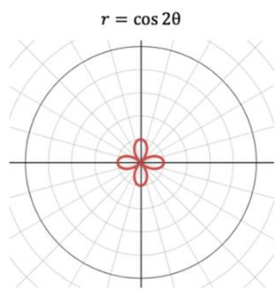
We have also learned that the point $(2, \frac{\pi}{2})$ can be written $(2, -\frac{3\pi}{2})$. The $-\frac{3\pi}{2}$ comes from $\frac{\pi}{2} - 2\pi$, which is $\frac{\pi}{2} - \frac{4\pi}{2} = -\frac{3\pi}{2}$. If the ray is rotated $270°$ clockwise at a distance of 2 units from the origin, it is located at the same position as the original point $(2, \frac{\pi}{2})$.

We also learned that r could be the opposite (or negative in this case), leading to two other names for the same point: $(-2, \frac{3\pi}{2})$ when moving counterclockwise to $270°$ and going in the opposite direction for $r = -2$, or $(-2, -\frac{\pi}{2})$ when moving clockwise from the pole and flipping to the opposite side for $r = -2$.

After investigating polar coordinates, we investigated polar equations that led to some very interesting polar graphs. We know the polar equation $r = 5$ is the graph of a circle with a radius of 5 that surrounds the pole. However, $r = \sin\theta$ is a circle with a diameter of 1 above the polar axis and $r = \cos\theta$ is a circle with a diameter of 1 to the right on the polar axis.

As we continued our explorations, we discovered that $r = \sin 2\theta$ creates a flower with four petals. The equation $r = \cos 2\theta$ does as well, but it is rotated from the graph of $r = \sin 2\theta$.





The graphs of $r = \sin 3\theta$ and $r = \cos 5\theta$ create flowers with three petals and five petals respectively (the same number as the coefficient of theta when it is an odd integer).

Looking Ahead 4.10

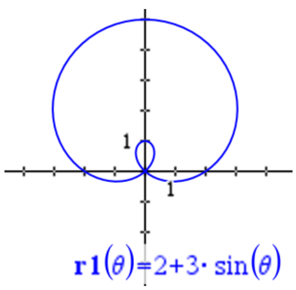
In this section, we will continue to investigate the designs created by polar equations. They are quite beautiful and quite complex. Therefore, we will be using the graphing calculator to create them. Some of them have special names.

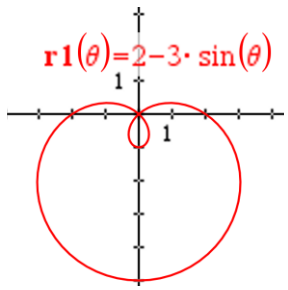
Example 1:	Graph the polar equation $r = \frac{3}{\tan 4\theta}$ on your calculator. What are some of your observations.

The number of designs in the universe God has created is without end!

Example 2: Graph the polar equation $r = \frac{5}{2\sin\theta + 3\cos\theta}$ using the graphing calculator. What are some of your observations.

Example 3: Limaçons are of the form $r = a + b\sin\theta$ and $r = a + b\cos\theta$ in which $a \neq b$. Graph the polar equation $r = 2 + 3\sin\theta$ and describe its features. Describe the longest length of the inner petal and outer petal in terms of a and b. On the same graph, graph the polar equation $r = 2 - 3\sin\theta$.





The length of the inner petal for $r = 2 + 3\sin\theta$ is 1.

$a = 2$ and $b = 3$

Therefore, 1 is $3 - 2$ or $|2 - 3|$.

The length of the inner petal for $r = 2 + 3\sin\theta$ is 1, which is $3 - 2$.

The length of the inner petal for $r = 3 + 7\sin\theta$ is 4.

$\mathrm{a} = 3$ and $\mathrm{b} = 7$

Therefore, 4 is $7 - 3$ or $|3 - 7|$.

The length of the inner petal for $r = 2 + 5\sin\theta$ is 3 from $5 - 2$ or $|2 - 5|$.

The length of the inner petal is $\mathrm{b} - \mathrm{a}$ or $|\mathrm{a} - \mathrm{b}|$ when $\mathrm{b} > \mathrm{a}$.

The length of the outer petal for $r = 2 + 3\sin\theta$ is 5, which is $2 + 3$.

The length of the outer petal for $r = 3 + 7\sin\theta$ is 10, which is $3 + 7$.

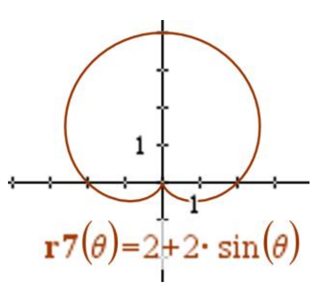
The length of the outer petal for $r = 2 + 5\sin\theta$ is 7, which is $2 + 5$.

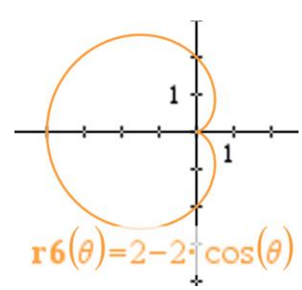
The length of the outer petal is $\mathrm{a} + \mathrm{b}$ or $|\mathrm{a} + \mathrm{b}|$.

The limaçons that have the sine function will be above the horizontal axis if there is a plus sign between a and b; the limaçons that have the sine function will be below the horizontal axis if there is a minus sign between a and b.

The limaçons that have the cosine function will be to the right of the vertical axis if there is a plus sign between a and b; the limaçons that have the cosine function will be to the left of the vertical axis if there is a minus sign between a and b.

The ratio of $\frac{\mathrm{a}}{\mathrm{b}}$ determines the exact shape of the limaçon. If $\mathrm{a} = \mathrm{b}$, the special limaçon is called a cartioid.





Section 4.11 Complex Numbers and the Complex Plane

Looking Back 4.11

In Problem 7 of Practice Problems 4.7, we converted the rectangular-coordinate equation $x^2 + y^2 = -5$ to the polar equation $r = \pm i\sqrt{5}$. The numbers $i\sqrt{5}$ and $-i\sqrt{5}$ are imaginary numbers. We learned about these in Algebra 2.

Before that, we worked mostly with real numbers. A complex number is a combination of real and imaginary numbers of the form a + bi in which a and b are real numbers and i is an imaginary number. The number "a" is the "real" part, and "bi" is considered the "imaginary" part. If b $= 0$, then we have only a real part, so the otherwise complex number reduces to a real one.

The definition of i is: $i = \sqrt{-1}$ and $i^2 = -1$. That is why $\sqrt{-5} = \sqrt{-1 \cdot 5} = \sqrt{i^2 \cdot 5} = i\sqrt{5}$. If r is a positive real number, then $\sqrt{-r} = i\sqrt{r}$.

Complex products may be simplified to real numbers: $\sqrt{-16} \cdot \sqrt{-36} = 4\sqrt{-1} \cdot 6\sqrt{-1} = 4\sqrt{i^2} \cdot 6\sqrt{i^2} = 4i \cdot 6i = (4)(6)(i)(i) = 24i^2 = 24(-1) = -24$.

As you may recall, complex numbers may be added, subtracted, multiplied, divided, and simplified using previously learned properties.

In Examples 1-4 that follow, simplify the complex expressions.

Example 1:	Simplify $\sqrt{-9y^2} + \sqrt{-25y^2}$. Just find the principal (positive) square root.

Example 2:	Simplify $\sqrt{-49\text{m}^4\text{n}^4} - \sqrt{-16\text{m}^4\text{n}^4}$. Just find the principal (positive) square root.

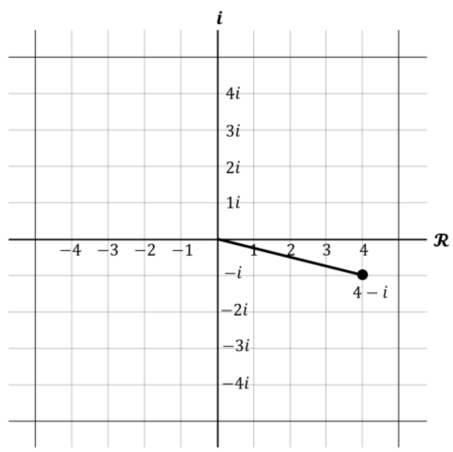
Example 3:	Simplify $\left(\frac{2}{3i}\right)\left(\frac{1}{2i}\right)$. Just find the principal (positive) square root.

Example 4: $\frac{8}{\sqrt{-2}}$

Looking Ahead 4.11

The complex plane consists of a real part ($\mathcal{R}$) and an imaginary part (i). The $\mathcal{R}$ (real) part is the horizontal component and the i (imaginary) part is the vertical component.

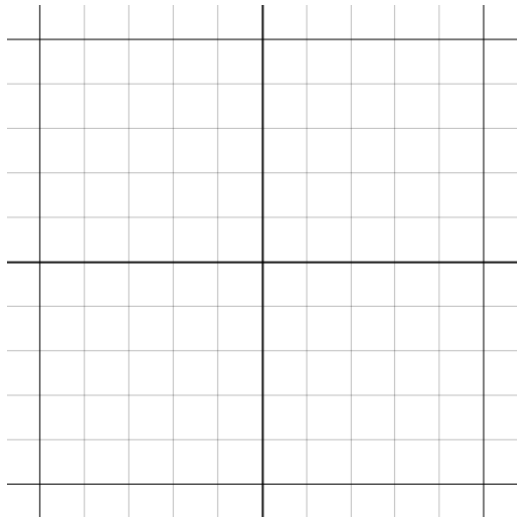
The complex number $z = \mathrm{a} + \mathrm{b}i$ can be plotted as the point (a, b) on the complex plane where "a" is the "real" part or the horizontal component and "b" is the "imaginary" part or the vertical component.



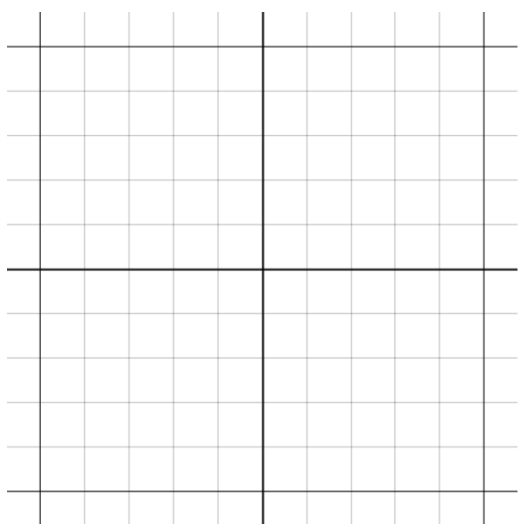
Note: The distance of the point $4 - i$ from the origin is the hypotenuse of the right triangle formed. The Pythagorean Theorem can be used to find the hypotenuse of this right triangle.

The complex number $z = 4 - i$ can be plotted as $(4, -1)$. The length of the line segment plotted between z and the origin is $\sqrt{4^2 + (-1)^2} = \sqrt{16 + 1} = \sqrt{17}$. This is also called the modulus or the magnitude of a complex number and is usually written $z = |\mathrm{a} + \mathrm{b}i| = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$. If the notation reminds you of the absolute value of a real number, that is no coincidence; $|z| = \sqrt{a^2} = |a|$ reduces to a standard absolute value if $z = a$ is real. Moreover, even when z is complex, $|z|$ gives the distance of the point representing z from the origin, just like $|a|$ gives the distance between the point representing 0 and a on the number line for a number a. Additionally, just like the standard absolute value, the complex modulus cannot be negative, and it only equals zero if z itself is zero. To emphasize the differences between the complex modulus and the standard absolute value, it is common to write the complex modulus with two vertical lines like so: $||z||$.

Example 5: Find the sum of $(2+6i)$ and $(-3+4i)$ and plot it on the complex plane.



Example 6: Find the difference of $(2+6i)$ and $(-3+4i)$ and plot it on the complex plane.



Section 4.12 Polar Form of Complex Numbers

Looking Back 4.12

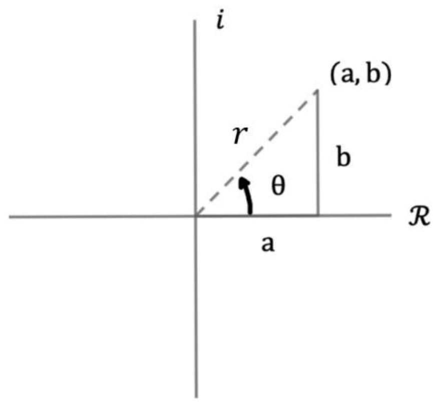
We are told in the Bible that God's thoughts are higher than our thoughts and God's ways are higher than our ways. Therefore, there are concepts in mathematics (such as infinity) which are difficult to comprehend. Imaginary numbers and complex numbers can also be difficult to grasp. They were invented to help us understand the root of a negative integer when the properties of square roots and real numbers tells us that multiplying two positive, or two negative numbers always yields a positive number.

In Pre-Algebra, we used dot-grid paper to understand the concept of irrational square roots. Using the polar plane is helpful in understanding complex numbers.

Example 1:	Use the Argand grid and the Cartesian grid to find the location of the roots of $y = x^2 - 2x + 3$.

<u>Looking Ahead 4.12</u>

Complex numbers, like rectangular numbers, can be converted to polar form or trigonometric form. Trigonometric form can be helpful in finding powers and roots of complex numbers (we have discussed this previously).



Polar coordinates in the rectangular plane are $x = r\cos\theta$ and $y = r\sin\theta$. Substituting a and b of the complex plane in for x and y yields the polar coordinates in the complex plane: $\mathrm{a} = r\cos\theta$ and $\mathrm{b} = r\sin\theta$. In this relationship, the polar form of a complex number becomes the following:

$$z = \mathrm{a} + \mathrm{b}i \qquad \text{or} \qquad z = r\cos\theta + r\sin\theta\, i$$

$$z = r(\cos\theta + i\sin\theta)$$

The distance of r is $\sqrt{\mathrm{a}^2 + \mathrm{b}^2}$. To find the angle θ when a and b are known from the complex number, use the trigonometric form $\tan\theta = \frac{\mathrm{b}}{\mathrm{a}}$.

Try using the real and imaginary parts of the polar coordinates of the point to write a complex number in polar form for the following example:

Example 2:	Convert the complex number below to polar form.

$$z = 7 + 7i$$

Example 3: Convert the complex number in polar form below to a complex number in standard form.

$$z = 2(\cos\left(\frac{\pi}{6}\right) + i\ \sin(\frac{\pi}{6}))$$

Example 4: Multiply $\cos(\frac{\pi}{2}) + i\ \sin(\frac{\pi}{2})$ by its complex conjugate and simplify the expression. You learned about complex conjugates in Algebra 2.

Section 4.13 Finding Roots and Powers of Complex Numbers

Looking Back 4.13

In Algebra 2, we learned the polynomial $f(x) = x^4 - 1$ has four roots because it is of degree 4. We know two of the roots are 1 and -1 because $f(1) = 1^4 - 1 = 1 - 1 = 0$ and $f(-1) = (-1)^4 - 1 = 1 - 1 = 0$.

Because $f(x) = 0$ when $x = 1$ and $f(x) = 0$ when $x = -1$, 1 and -1 are two roots and $(x - 1)(x + 1)$ are two of the factors. This is equal to $x^2 - 1$, which is a difference of squares. If we divide $x^4 - 1$ by $x^2 - 1$, we can find the other factor.

Example 1:	Use long multiplication to find the other factor of the polynomial $f(x) = x^4 - 1$.

$$x^4 - 1 = (x^2 - 1)(x^2 + 1)$$

To find the zeroes, set the factored form equal to 0.

$x^2 - 1 = 0$ or $x^2 + 1 = 0$

$x^2 = 1$ $x^2 = -1$

$x = \pm\sqrt{1}$ $x = \pm\sqrt{-1}$

$x = \pm 1$ $x = \pm i$

These are the two roots previously found. These are two other roots for a total of four roots.

<u>Looking Ahead 4.13</u>

Finding roots and powers of complex numbers can be a daunting task but is necessary to solve some problems in Calculus.

First, we must decipher how to multiply two complex numbers in polar form.

Example 2: Multiply the following: $z_1 = r_1(\cos \mathrm{a} + i \sin \mathrm{a})$ and $z_2 = r_2(\cos \mathrm{b} + i\ \sin \mathrm{b})$.

$z_1 \cdot z_2 = r_1(\cos \mathrm{a} + i\ \sin \mathrm{a}) \cdot r_2(\cos \mathrm{b} + i\ \sin \mathrm{b})$

$= r_1 \cdot r_2[(\cos \mathrm{a} + i\ \sin \mathrm{a}) \cdot (\cos \mathrm{b} + i\ \sin \mathrm{b})]$ Commutative Property of Multiplication

$= r_1 \cdot r_2[(\cos \mathrm{a}\ \cos \mathrm{b} + \cos \mathrm{a}\ i \sin \mathrm{b} + \cos \mathrm{b}\, i \sin \mathrm{a} + i^2 \sin \mathrm{a} \sin \mathrm{b})]$ Distributive Property

$= r_1 \cdot r_2[(\cos \mathrm{a}\ \cos \mathrm{b} + \cos \mathrm{a}\ i \sin \mathrm{b} + \cos \mathrm{b}\, i \sin \mathrm{a} - \sin \mathrm{a} \sin \mathrm{b})]$ Because $i^2 = -1$

$= r_1 \cdot r_2[(\cos \mathrm{a} \cos \mathrm{b} - \sin \mathrm{a} \sin \mathrm{b}) + i(\sin \mathrm{a} \cos \mathrm{b} + \cos \mathrm{a} \sin \mathrm{b})]$

$= r_1 \cdot r_2[\cos(\mathrm{a} + \mathrm{b}) + i \sin(\mathrm{a} + \mathrm{b})]$ Sum and Difference Properties for Sine and Cosine

Example 3: Let $z_1 = 2(\cos 45° + i \sin 30°)$ and let $z_2 = 3(\cos 15° + i \sin 30°)$. Find the product of $z_1 \cdot z_2$.

$z_1 z_2 = 2 \cdot 3[\cos(45° + 15°) + i\ \sin(30° + 30°)]$

$= 6[\cos 60° + i\ \sin 60°]$

A shortcut can be used for polar products: Multiply the moduli: $|z| = r$, and add theta (θ), which is called the argument of z and can be in degrees or radians.

The quotient of two polar numbers now follows:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\ [\cos(\mathrm{a} - \mathrm{b}) + i\ \sin(\mathrm{a} - \mathrm{b})]$$

A shortcut can be used for polar quotients:
Divide the moduli: $|z| = r$, and subtract the arguments.

Example 4: Now we can use the product formula to find powers of complex numbers in polar form. Find z^2 in polar form.

Perhaps you see another pattern in God's glorious mathematical creations as did Huguenot Moivre. Moivre was born in Paris and educated at a protestant school, Academy of Sedan, where he studied Greek. This school was forced to close in 1681 due to its profession of faith.

Moivre later moved to England where he tutored nobility in games of chance and changed his name to Abraham De Moivre. He was a follower of the works of Isaac Newton and like Newton, published his own works: for one, an article called "The Doctrine of Chances." Moivre took a great interest in loans, mortgages, pensions, and other practical applications of mathematics as well.

Of Moivre, Ivo Schneider (a modern mathematician) writes:

> Since the supreme goal of scientific enterprise is to demonstrate the existence of an agent, called God, whose constant activity permeates the whole cosmos, de Moivre with his interplay between law and design, which reveals the existence of God, and chance, which represents His constant activity, had reached this goal. *

De Moivre's Theorem states that "If $z = r(\cos\theta + i\ \sin\theta)$ in complex polar form and n is a positive integer, then $z^n = r^n\ (\cos n\,\theta + i\ \sin n\,\theta)$."

* *"Statisticians of the Centuries," C.C. Heyde and E. Seneta Editors, 2001 Springer Science + Business Media, New York*

Example 4:	Use De Moivre's Theorem to find z^4 in a + bi form. Let z equal $1 + i\sqrt{3}$.