Geometry and Trigonometry Module 4 Polygons and Quadrilaterals

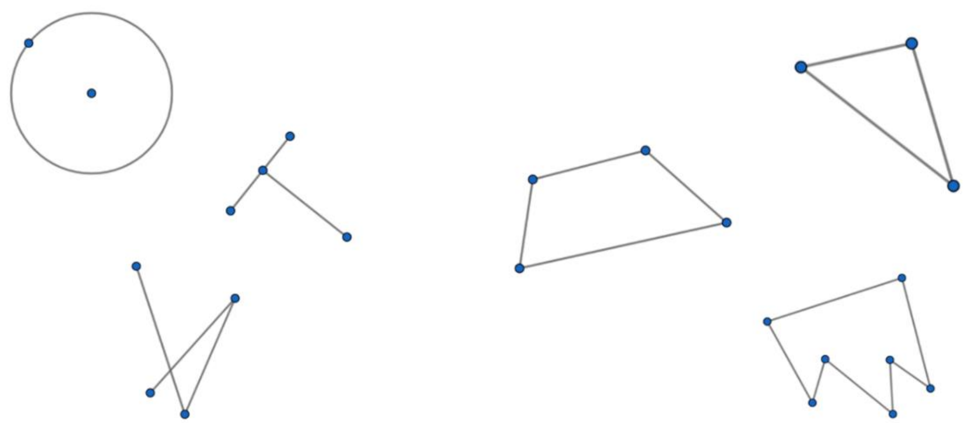
Section 4.1 Classifying Polygons

Looking Back 4.1

In Module 3, angles and angle relationships were explored in the introduction to geometry. In this module, polygons will be investigated. The next module is all about circles. Module 6 will be entirely devoted to triangles because of their strong connection to trigonometry. We will close *Geometry and Trigonometry* and more with two modules of an overview of trigonometry.

A polygon is a many-sided figure; however, they are flat (2-dimensional). The Greek word ‘poly’ means many and ‘gon’ comes from the Greek word ‘gonu,’ which means knee (and was translated into the English word ‘angles’). A polygon has many angles and angles are made up of sides (at least two). Therefore, a polygon is a figure with many angles and many sides. Moreover, it is a closed figure of line segments connected endpoint to endpoint. Each line segment is called a side and is connected to two other sides on either end of the segment. Just as with angles, the endpoint where the sides meet is called a vertex. Since a polygon has many angles, it also has many vertices. A polygon is a closed figure that has sides made of line segments that meet at the endpoints and may not cross each other.

Example 1: Circle the figures below that are polygons. Put an ‘X’ through the figures that are not polygons.



Organisms are named based on classification: the same concept applies to geometry. All polygons are named based on classification by the number of sides they have. Below is a list of the polygons with their Greek root word underlined, the number of sides they have in the left column, and the meaning of the prefix root word in the right column. (This list could go on and on, but we will stop at 10.)

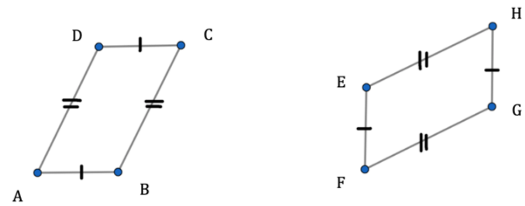
Number of Sides	Name	Prefix Root Word
3	Triangle	(three angles)
4	Quadrilateral	(four sides)
5	Pentagon	(five)
6	Hexagon	(six)
7	Heptagon	(seven)
8	Octagon	(eight)
9	Nonagon	(nine)
10	Decagon	(ten)

(Note: A regular polygon is a polygon that has all sides equal and all angles congruent.)

Looking Ahead 4.1

Now that polygons have been introduced and formally defined, it is time to explore them a bit more deeply. In Biology, questions are asked to determine further classifications for a given species. There are parts of polygons that are definitions, conjectures, postulates, or may lead to formal proofs.

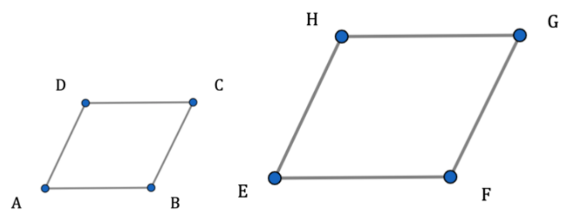
Congruent figures have the same size and shape.



ABCD $\cong$ FEHG

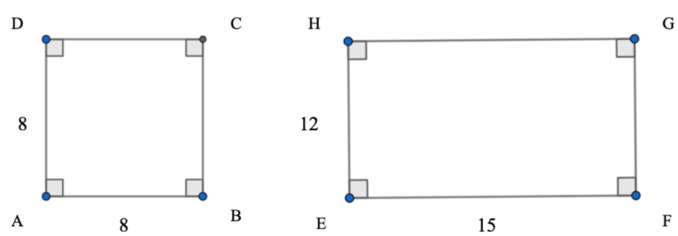
The lengths of corresponding sides are equal. The measures of corresponding angles are congruent.

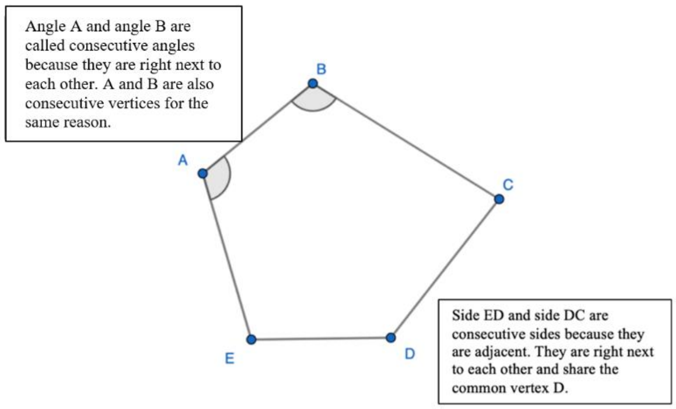
Figures are similar if they have the same shape, but not necessarily the same size. In the figure below, the corresponding angles are congruent, and the corresponding sides are proportional; the sides dilate (shrink) by a common ratio.



The lengths of the sides of parallelogram EFGH are 1.5 times longer than the lengths of the sides of parallelogram ABCD. The corresponding angles are still congruent: ABCD $\sim$ EFGH.

Example 2:	Is square ABCD below similar to rectangle EFGH?

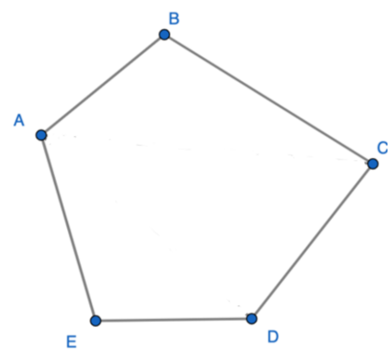




A polygon has consecutive sides on either side of the endpoint. It also has consecutive angles on either side of an angle at the vertices. A polygon has consecutive vertices at these consecutive angles as well.

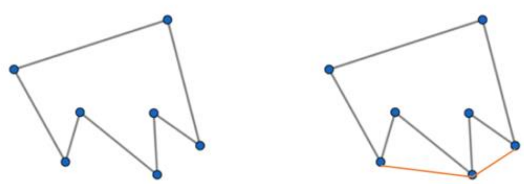
Consecutive sides and consecutive angles are next to each other. These are not to be confused with corresponding sides and corresponding angles, which compare two figures to determine if they are similar, congruent, or neither.

In the pentagon ABCDE to the right there are two consecutive angles, B and E, on either side next to angle A. The other two angles, D and C, are opposite angle A. They are called non-consecutive angles. The sides of angle D and angle C do not touch angle A.



Example 3:	What are the consecutive angles to angle D? What are the opposite angles to angle D?

The line segment from any vertex to a non-consecutive vertex in a polygon is called a diagonal. All the diagonals that can be drawn in pentagon ABCDE above are inside the figure; therefore, pentagon ABCDE is called a convex polygon. The upside-down crown shown below is a concave polygon because two of the diagonals, shown in orange, are outside of the polygon.



Section 4.2 Classifying Quadrilaterals

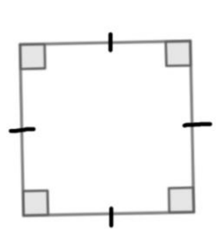
Looking Back 4.2

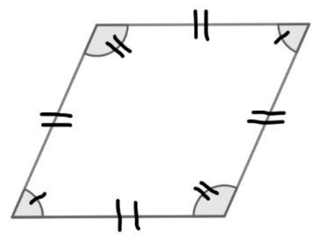
The polygon we will investigate in this section is a quadrilateral. There are many different types of quadrilaterals, which are classified depending on their properties. Though they will be introduced in this section, each quadrilateral will be explored in more depth in upcoming sections.

All quadrilaterals have four sides since "quad" means four and "lateral" means side. A quadrilateral is a 2-dimensional (flat) shape with straight lines for sides that are joined. Moreover, a quadrilateral is a closed four-sided figure.

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

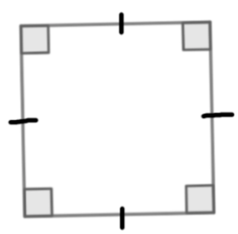
A parallelogram that has four congruent sides and four right angles is a square (below to the left). A parallelogram that has four congruent sides is a rhombus (below to the right).

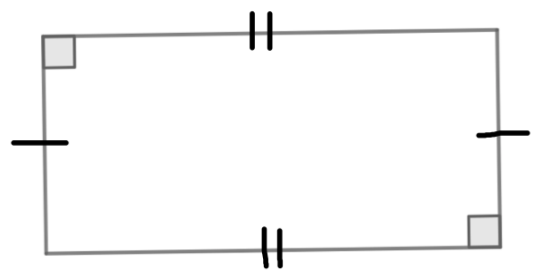




A parallelogram that has four right angles is a rectangle.

What is similar in the rectangle and square shown below? What is different? Is a square always a rectangle or is a rectangle always a square?

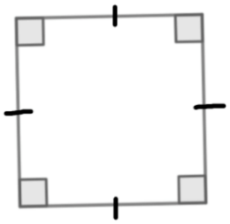




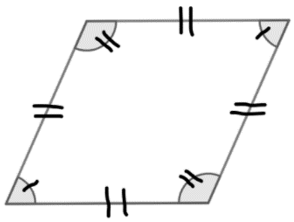
Both the square and the rectangle above have four right angles, so the angles are all equal (90°). In each figure, the adjacent sides are perpendicular and both pairs of sides are parallel. However, all sides are congruent in the square (which means opposite sides are equal), but not in the rectangle. Therefore, a square is a special case of a rectangle, but a rectangle is not a square because all the properties of a square are not true for a rectangle.

What other properties do you notice in a parallelogram?

Example 1:	Draw the lines of symmetry through the square. How many are there? Are all squares similar?

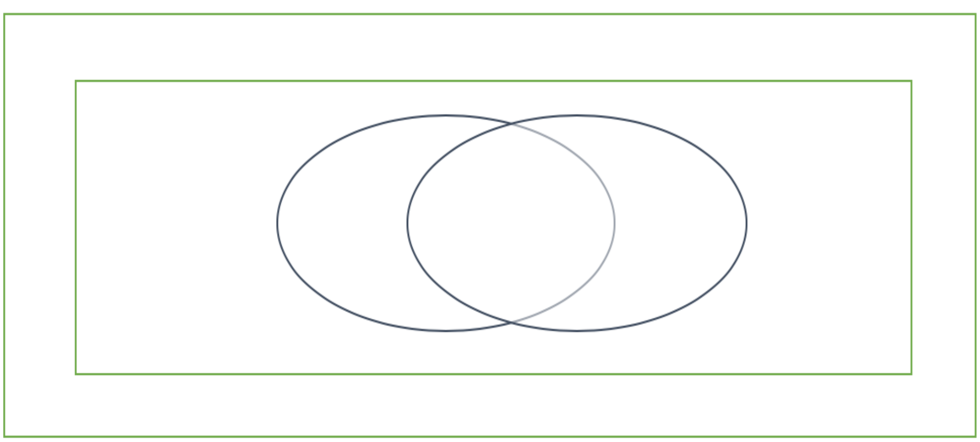


Example 2:	Draw the lines of symmetry for the rhombus. How many are there? Are all rhombi similar?



In a square, all the angles are equal and each angle is 90° degrees. All four angles are right angles. In a rhombus, opposite angles are equal. One pair is obtuse and the other pair is acute. What the square and rhombus both have in common is that all sides are equal. Therefore, a rhombus is not a square because all the properties of a square are not true for a rhombus. Still, all the properties of a rhombus hold true for a square, so we say that a square is a special case of rhombus.

Example 3:	Complete the Venn diagram below to show the relationship between the rectangle, square, and rhombus.



Section 4.3 Angle Sums of Polygons

Looking Back 4.3

In the first section of this module, you were shown a chart with all the names of the polygons from shapes with 3 sides to shapes with 10 sides. In this section, you will investigate the sum of all interior angles in several polygons from that chart. As previously stated, a regular polygon has all sides equal and all angles congruent.

A triangle with all sides and all angles equal is an equilateral triangle. We know the sum of the angles of a triangle is $180°$ degrees; therefore, in an equilateral triangle (which is equiangular) each angle must be $60°$ since $180° \div 3 = 60°$ degrees.

A quadrilateral with all sides equal and all angles equal is a square. We know the sum of the angles of a quadrilateral is $360°$ since it can be cut into two equal triangles ($180° \cdot 2 = 360°$); therefore, in a square, which has four equal angles, each angle must be $90°$ since $360 \div 4 = 90°$ degrees. The corners of a square are right angles so each one is $90°$ degrees.

Looking Ahead 4.3

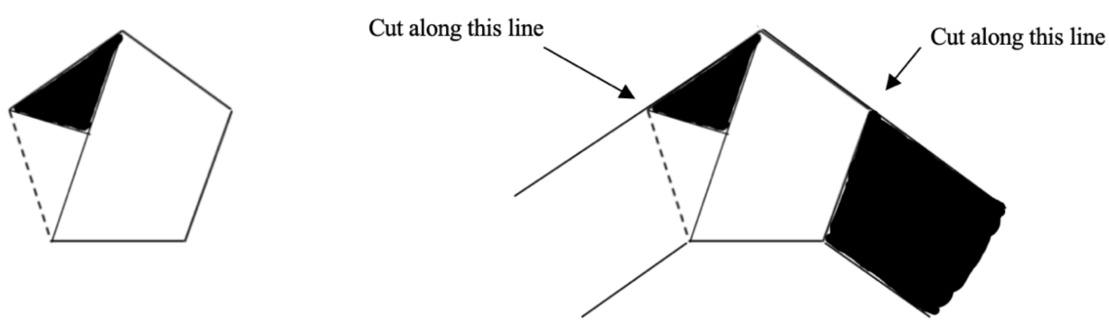
Since we know the sum of the angles of a triangle, we can break regular polygons up into triangles to find the sum of the angles of the polygon. We can divide this sum by the number of angles (since they are all equal) to find the degree of each angle.

Let us start with a pentagon. You can use a long strip of paper with a constant width to fold a pentagon. Lay a ruler along the edge of the paper and mark it to get the constant width. To tie a paper knot pentagon:

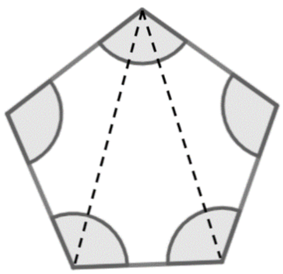
1) Tie an overhand knot as if you are tying your shoes. It looks like a pretzel.

2) Tighten it and flatten the creases.

3) Cut off the lengths hanging out from the sides of the pentagon.



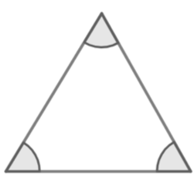
Example 1:	What is the sum of the interior angles of a regular pentagon? What is the measure of each angle?

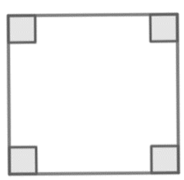


If you connect the vertex to the adjacent vertices, it simply forms the sides of the polygon. So, if a polygon has n sides it can be sliced into $n-2$ triangles. Since the angles of the triangle add up to $180°$ degrees, the angles of the polygon add up to $180°(n-2)$.

To find the degree of each angle, divide by the number of angles, which is equal to the number of sides of the polygon, n. The sum of the interior angles of a polygon is: $180°(n-2)$. The degree of each interior angle is: $\frac{180°(n-2)}{n}$.

Example 2:	What is the formula for the sum of the exterior angles of an n-gon? What is the formula for each exterior angle of an n-gon?





Section 4.4 The Geometry of Tangrams

Looking Back 4.4

The Tangram is an ancient Chinese geometric puzzle that is made up of seven polygons. It has been used to solve problems of mathematical recreation and is believed to be the tool the Chinese used to solve the Pythagorean Theorem. However, this was before it was called the Pythagorean Theorem, because although it was named after Pythagoras, the Chinese discovered it long before he was born. The Tangram can also be traced back to the third century to Archimedes, who used a Tangram-like puzzle to solve mathematics problems. In the practice problems section, we will further investigate how Tangrams were used to discover the Pythagorean Theorem.

One Tangram piece appeared in a book as early as 1742, but the first book of Tangrams appeared around 1813. This date is not entirely reliable; however, by 1817 we know these books appeared in the United States and Europe as well.

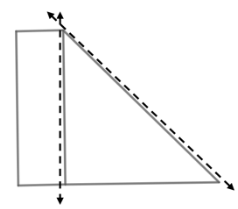
For a written work to be reliable, it must pass a bibliographic test and internal and external evidence tests. The Bible fulfills this. It is historically, geographically, and archaeologically accurate and claims to be the infused and inspired Word of God the Creator of all it contains. You can investigate this on your own.

But first, we will begin this section by making our own set of Tangrams and then using them to solve the problems.

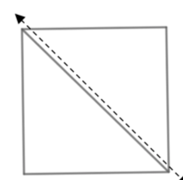
Looking Ahead 4.4

Begin with a piece of blank white paper to make the seven Tangram puzzle pieces.

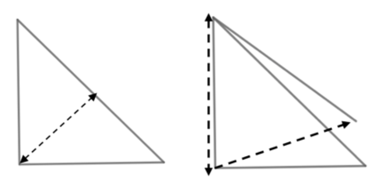
1. Lay the 8.5” by 11.5” piece of paper so the long side is on the bottom and the short side is on the side. Fold the top right corner down until the edge lines up with the bottom of the piece of paper. Cut off the excess rectangle on the left.



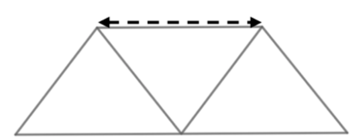
2. Open the remaining piece and you will have a square with a diagonal through it (the diagonal connects two opposite vertices). Cut along the diagonal so that you have two large congruent triangles.



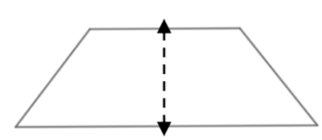
3. Fold one large triangle along the median (the fold line will be from the right angle perpendicular to the opposite side). Cut along the fold, creating two more medium congruent triangles.



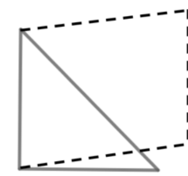
4. Lay the other large triangle down so that one vertex is pointing up and one side is on the right, another is on the left, and the third side is the base. Fold the top vertex over onto the opposite side, meeting the base at the midpoint, and fold along the top. Open it back up and cut off the top triangle where the fold is.



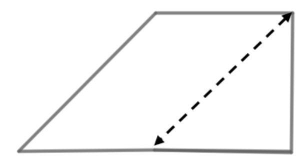
5. Fold the trapezoid in half from left to right and cut along the fold.



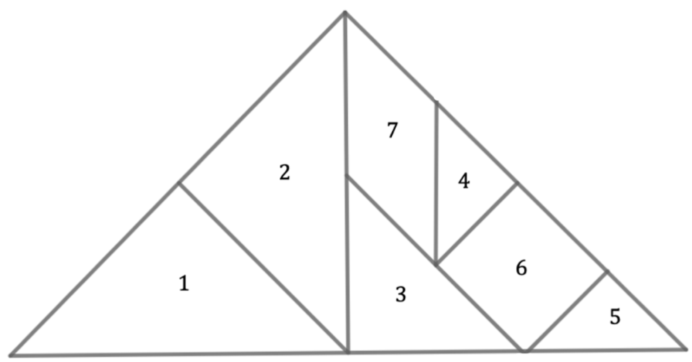
6. Take one of the congruent trapezoids and fold it so that a triangle lies on top of the square. This will be along the altitude from the top vertex perpendicular to the base. Cut along the altitude, creating a small triangle and a small square.



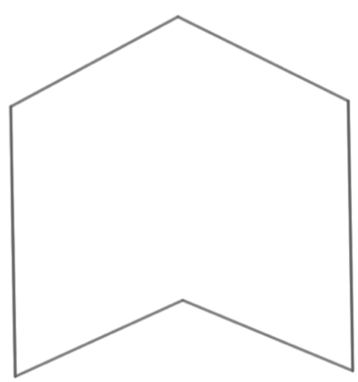
7. Fold the right angle of the other trapezoid up to the opposite vertex and fold. Cut along the fold, creating a parallelogram and another small triangle.



8. You now have two large congruent triangles, one medium triangle, and two small congruent triangles. Put the congruent triangles on top of one another to confirm they are the same size and shape. You also have one parallelogram and one small square. Use the puzzle pieces to make a large triangle. The solution is shown below, and each piece is numbered. Number each of your pieces the same to use in another activity.



Use just two pieces to make a large triangle that is smaller than the one shown above.
Use pieces 1 and 2 only to make a square.
Use pieces 1 and 2 only to make a parallelogram.
Use six of the pieces to make a square.
Try to make a parallelogram and a square using all seven pieces.
Solutions are shown at the end of the practice problems. Try to use all seven pieces to make the shape below.



Section 4.5 The Geometry of Star Polygons

Looking Back 4.5

Geometry is often used, in art particularly, when geometric patterns develop. We can see examples of this in the world. For one, the cathedrals in Italy have domes with mosaic designs that were created using geometry. We also see geometric patterns in nature, such as spirals and waves, that may be considered God's art.

We will view a direct example of geometric art when we investigate kaleidoscopes in the next section. Another previous example was when we made string art designs and even before in a previous course when we created irregular tessellations.

Looking Ahead 4.5

In this section, we will be creating star polygons. A star polygon is inscribed in a circle. Points are first placed equidistantly around the circle. The points of the star are made by connecting every mth point of the n points that divide the circumference of the circle into n equal parts. The star is called a non-simple polygon because the lines do cross as they are connected around the circle to form a star.

Let us start with a 5-star polygon. This means $n = 5$ and the circumference of the circle is divided into 5 equal parts as 5 points are placed on the circumference of the circle. Brackets are used to imply a star polygon is to be formed. In the brackets, the top number (n) is the number of points on the circle's circumference and the bottom number (m) represents which points are to be connected beginning clockwise or counterclockwise around the circle. If $m = 3$, every third point on the circumference is to be connected. The notation is shown below:

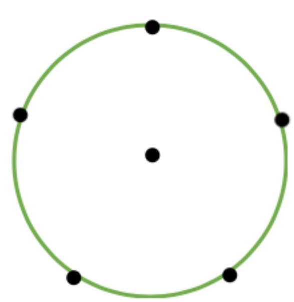
$$\begin{Bmatrix} n \\ m \end{Bmatrix}$$

So, there is no such thing as $\begin{Bmatrix} 5 \\ 0 \end{Bmatrix}$ because no points would be connected, and the circle would remain empty. Do you think there is such a thing as a $\begin{Bmatrix} 5 \\ 5 \end{Bmatrix}$-star polygon?

This means that the 5th point would continually be connected to itself. No star would be formed inside the circle.

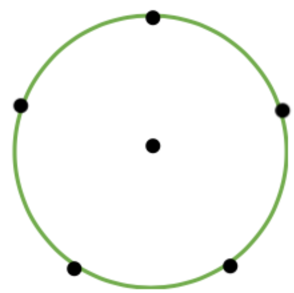
Let us try to find all the possible 5-star polygons. Excluding $\begin{Bmatrix} 5 \\ 5 \end{Bmatrix}$, our only options are to connect every 1st point, 2nd point, 3rd point, and 4th point.

In the figure below, start at the top point and connect it to the next point to the left. Continue this pattern around the circumference and you will get a pentagon for a $\begin{Bmatrix} 5 \\ 1 \end{Bmatrix}$ polygon.

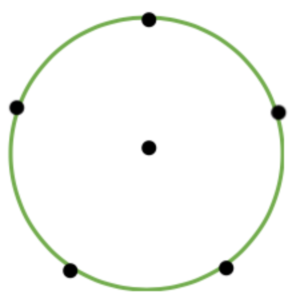


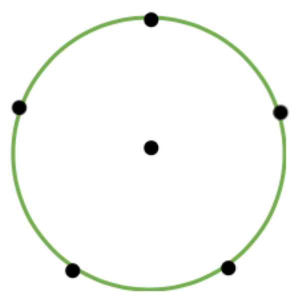
This is a pentagon. This is definitely a polygon, but not a star. When $m = 1$, the polygon will always have n sides and be named by its number of sides. Therefore, $m \neq 1$ and $m \neq 0$ for star polygons.

Now, let us try a $\left\{\begin{matrix}5\\2\end{matrix}\right\}$ star polygon. Start at 0 and move 2 points counterclockwise. Draw a line segment from 0 to 2. From 2, move another 2 points counterclockwise. Draw a line segment from 2 to 4. From 4, move another 2 points counterclockwise. Draw a line segment from 4 to 1. Keep repeating this pattern until you are drawing a line over an existing line. You will be back to where you started.



Draw a $\left\{\begin{matrix}5\\3\end{matrix}\right\}$ polygon and a $\left\{\begin{matrix}5\\4\end{matrix}\right\}$ polygon. What do you notice?





The $\left\{\begin{matrix}5\\3\end{matrix}\right\}$ is the same as the $\left\{\begin{matrix}5\\2\end{matrix}\right\}$ polygon and the $\left\{\begin{matrix}5\\4\end{matrix}\right\}$ is the same as the $\left\{\begin{matrix}5\\1\end{matrix}\right\}$ polygon. These are called compliments. If $\left\{\begin{matrix}n\\m\end{matrix}\right\} = \left\{\begin{matrix}5\\1\end{matrix}\right\}$ and its compliment is $\left\{\begin{matrix}5\\4\end{matrix}\right\}$, how would the compliments be written using star polygon notation in terms of n and m?

It would be written $\left\{\begin{matrix}n\\m\end{matrix}\right\} = \left\{\begin{matrix}n\\n-m\end{matrix}\right\}$.

There are only 2 unique 5-star polygons that can be formed.

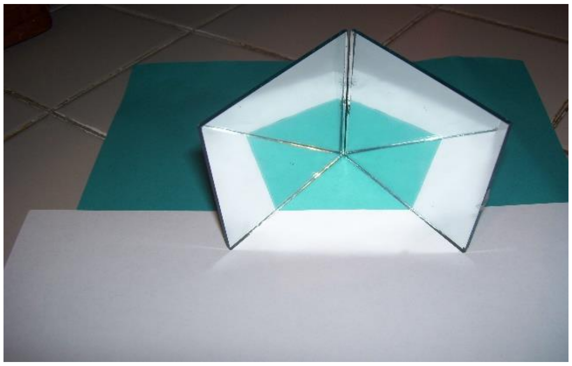
Section 4.6 Central Angles and Apothems

Looking Back 4.6

In the past, we have used different methods to investigate kaleidoscopes. Today, we are going to use mirrors to explore what the inside of a kaleidoscope looks like and learn about central angles.

In the last section, we saw stars inscribed in circles. Today, the mirrors will help us view the central angle in a polygon. The circle it is inscribed in would be the end of the kaleidoscope.

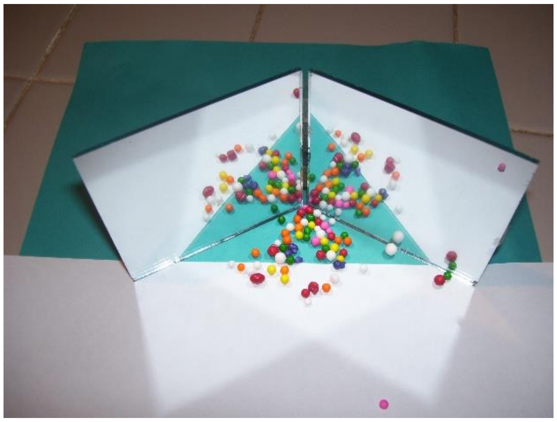
In order to use mirrors to investigate how kaleidoscopes work, we will begin by gathering some materials: two small square mirrors, a sheet of colored paper, a sheet of white paper, and tape. Place the materials on a surface. Put the colored piece of paper down so it is overlapping the piece of white paper. Now, tape the mirrors together at an edge so you can open the now adjoined mirrors at an angle; this is the swing angle. Place the mirror so it is standing on the paper as shown below, half on the colored paper and half on the white paper. As you open and close the mirror, you will see colored polygons inside it. The polygons are broken into several triangles that all meet in the center. This is the central angle. We cannot go inside the mirror to measure it, but we can put a protractor on top of the two mirrors to get the angle of the opening, which is equal to the central angle.



The mirror is open wide enough to see the pentagon in color. Placing a protractor on top of the mirrors to measure the swing angle shows approximately $72°$ degrees. That makes sense. If we multiply $72°$ by the five angles in the center of the pentagon, we get $360°$ degrees. There are $360°$ in a circle and this pentagon is inscribed in the circle of our kaleidoscope.

Looking Ahead 4.6

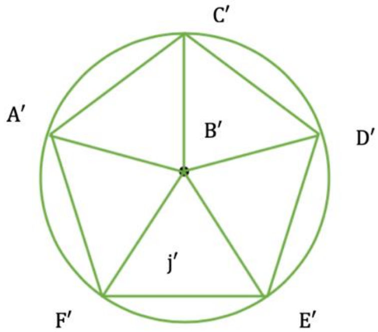
Referring to the previous activity, how could you find the measure of the central angle of the triangle in the kaleidoscope? I put confetti on the paper so you can see how a kaleidoscope works: the three central mirrors in the tube connect to form a triangle and reflect the images in the center onto one another.



Is the swing angle wider or narrower than the pentagon?

This one is wider. There are only three congruent triangles that meet at the central angle. They must each be $120°$ since $120° \cdot 3 = 360°$ degrees. Using reverse thinking, $360° \div 3 = 120°$ degrees. If the figure has n sides then the formula to find the central angle is $360° \div n$. The number of triangles formed in the center is equal to the number of sides. This only works for regular polygons that have all sides equal. Since mirrors are used to reflect the lines, they are all equal.

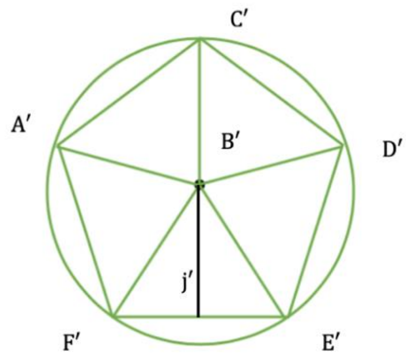
When a regular polygon is inscribed in a circle, the vertices of the polygon lie on the circumference of the circle. Line segments can be extended from each vertex to meet in the middle of the circle. These divide the polygon into equivalent isosceles triangles and form central angles that are also congruent to one another.



Now, before we move on, keep your mirrors out because you will be finding polygons with from 3 to 10 sides and completing a chart to find the measure of the swing angle in the practice problems.

Our lives can be mirrors to reflect the life of Christ and Scripture tells us to do just that. We are to embody love, peace, patience, and kindness just as Jesus Christ did when he was on earth. We are to be doers of the Word of God. James 1:23 says, “For if anyone is a hearer of the word and not a doer, he is like a man who looks at his natural face in a mirror; for once he has looked at himself and gone away, he has immediately forgotten what kind of person he was. But one who looks intently at the perfect law, the law of liberty, and abides by it, not having become a forgetful hearer but an effectual doer, this man shall be blessed in what he does.”

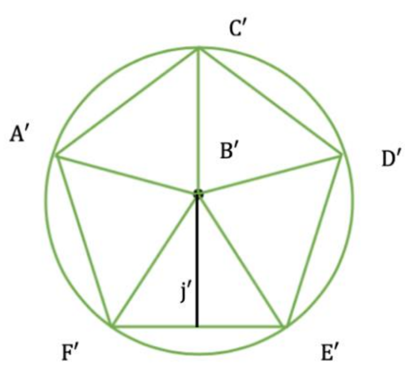
Now, we know the formula for the area of a triangle is $\frac{1}{2}b \cdot h$. In the pentagon above, the base of each of the five triangles is the length of the side of the pentagon. The height must be constructed so it can be measured. It is the perpendicular line segment from a vertex to its base on the opposite side. In the pentagon below, the top vertex of each triangle is where the central angles meet. The perpendicular line segments can be drawn to each base. The bases are the sides of the pentagon. The dark black line shown below is called the apothem because it comes from the central angle of the polygon inscribed in the circle and is perpendicular to the base of the triangle.



Our formula for area, which was $\frac{1}{2}b \cdot h$, has now become $\frac{1}{2}s \cdot a$, where s represents the side of the pentagon and a represents the apothem. This is only the area for one triangle, but in a pentagon, there are five congruent triangles. To get the area of the pentagon, the area of one triangle must be multiplied by 5, since there are five sides of a pentagon. It must be multiplied by 3 for a triangle, 5 for a pentagon, and n for an n-gon. The formula to find the area of a regular polygon inscribed in a circle is shown as follows:

$$\text{Area} = \frac{1}{2}s \cdot a \cdot n$$

Example 1: Find the area of the pentagon if the side length is s = 2.74 cm. and the radius of the circle circumscribed about the pentagon has a radius of r = 4 cm. Find the apothem first.



Note: Since Area $= \frac{1}{2}s \cdot a \cdot n$ is the same as Area $= \frac{1}{2}a \cdot s \cdot n$ and $s \cdot n$ is the perimeter of the polygon, the formula is sometimes written:

$$\text{Area} = \frac{1}{2}a \cdot P$$

Section 4.7 Areas of Polygons

Looking Back 4.7

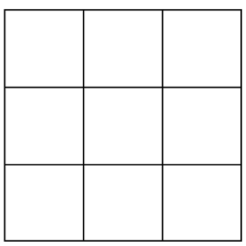
In the last section, you learned the area of a polygon with five or more sides could be found using the formula: $\mathrm{A} = \frac{1}{2}s \cdot a \cdot n$, where s is the length of the side of the polygon, a is the apothem, and n is the number of sides. Since multiplication is commutative and since $s \cdot n$ is the perimeter of the polygon, then P can be substituted into the formula for $s \cdot n$, making the formula: $\mathrm{A} = \frac{1}{2}a\mathrm{P}$.

The area of a regular polygon is given by the formula $\mathrm{A} = \frac{1}{2}\, a\mathrm{P}$. The polygon does not need to be inscribed in a circle. We simply did that so we could use our prior knowledge to derive the formula for area by comparing it to known measures.

In this section, we will be reviewing the areas of many polygons we already know and deriving formulas for new ones based on that prior knowledge of those we already know.

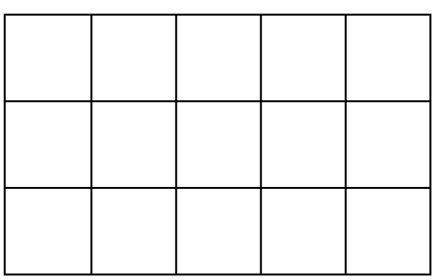
By now, you have learned that to get the number of square units in a square, one must add up all the unit squares inside the square or multiply the number of square units in each row by the number of square units in each column. The length and width are congruent in a square, so the formula is: $\mathrm{A} = s^2$, where s represents the length of each side.

Example 1: Find the area of the square in square units.



Finding the area of a rectangle is much like finding the area of a square. It is done by adding up all the squares inside the figure or multiplying the number of squares in a row by the number of squares in a column. However, since the sides are not of equal length, the sides are called ‘length’ and ‘width.’ The length may represent the longer sides while the width represents the shorter sides. The formula for the area of a rectangle is $\mathrm{A} = l \cdot w$.

Example 2: Find the area of the rectangle in square units.

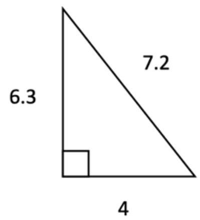


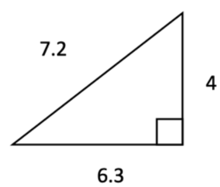
Since area is 2-dimensional, square units or units^2 are acceptable labels. Here, we do not know if the units are in centimeters, inches, or yards; we do not know the length of one unit.

Since we can draw a diagonal through a rectangle and get two triangles, the area for one triangle would be half the area of the rectangle. In a triangle, we do not use length and width, but rather base and height. This is because our height could be inside the triangle, outside the triangle, or along the side of the triangle. The formula for area of a triangle is: $A = \frac{1}{2}(b \cdot h)$.

In a right triangle, either leg could be the base and the other leg could be the height.

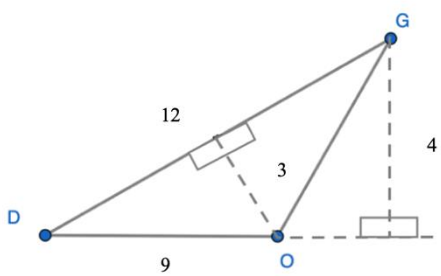
Example 3:	Find the area of the right triangles.





An altitude is a perpendicular segment drawn from any vertex to the opposite side in a polygon. The altitude can be drawn to a line that contains the opposite side of the polygon.

Example 4: Find the area of the obtuse triangle DOG using first the internal altitude and then the external altitude. Are both areas the same?



In the obtuse triangle DOG, the area can be found using the height inside the triangle from vertex O perpendicular to base DG. This is the internal altitude of the triangle. The height is 3 units and the base is 12 units. The area of triangle DOG is 18 square units.

Using the external altitude, a perpendicular line can be drawn from vertex G perpendicular to side DO when the line segment is extended. The height is now 4 units and the base is 9 units, so the area of triangle DOG is again 18 square units.

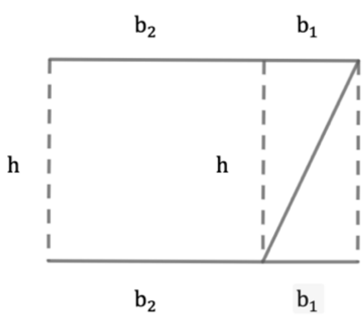
Looking Ahead 4.7

To find the area formulas for new polygons, compare them to what you already know about old polygons. For example, you used what you knew about the area of a rectangle to find the area of a triangle because you could compare the two.

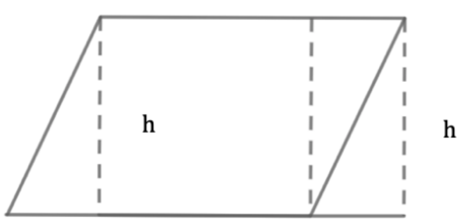
Now that we have reviewed the area of a triangle, let us use that information to find the area of a parallelogram.

Break the parallelogram into a rectangle with two equal triangles at both ends of the rectangle. Call the base of the triangles b_1 and call the base of the rectangle b_2. Call the height h. The height is the shared side of the rectangle and each triangle (there are two of them).

If you slide the triangle on the left side of the parallelogram over to the right side of the parallelogram, you can see that it creates a rectangle whose height is the altitude of the parallelogram (h) and whose base is the entire length of the parallelogram $(b_1 + b_2)$. So, the formula for the area of a parallelogram is also: $A = b \cdot h$.



Note: The base of the parallelogram is the length of the top/bottom parallel lines and the altitude can be measured inside or outside the parallelogram from one vertex perpendicular to the opposite side for the height.



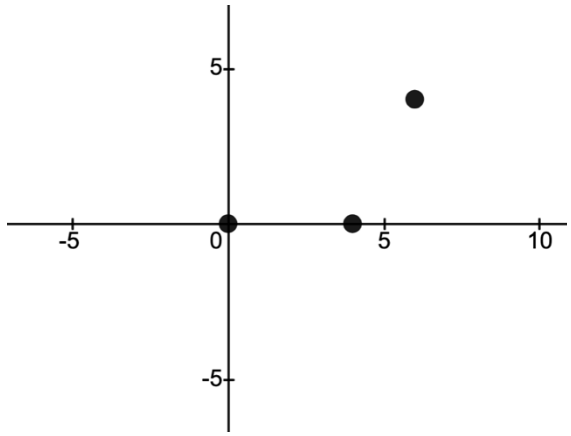
Example 5:	Find the base of the parallelogram if the area is 16.33 $cm.^2$ and the height is 2.3 cm.

More area formulas for trapezoids and kites will be derived in the practice problems.

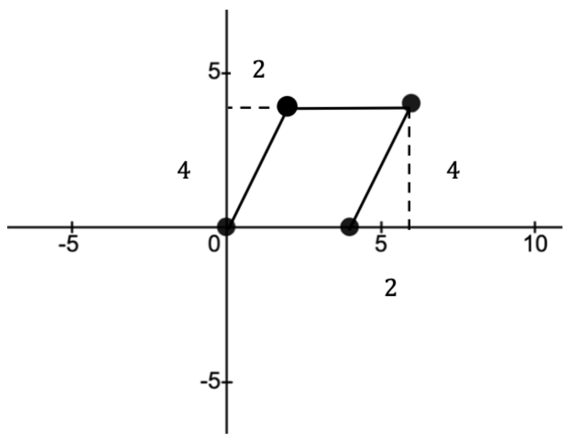
Section 4.8 Properties of Parallelograms

Looking Back 4.8

Suppose you are given the points (0, 0) and (4, 0). If a third point is located at (6, 4), where is the fourth point located that will form a parallelogram that has opposite sides parallel and congruent?



The distance of the bottom base is 4 units, so the distance of the top base must also be 4 units. That makes the location of the fourth point (2, 4). Now, we can form triangles on both sides of the parallelogram and use the Pythagorean Theorem to find the length of each side.



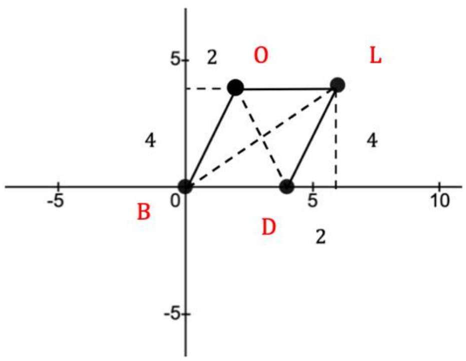
The distance of the right side of the parallelogram is the hypotenuse of the triangle on the right of the figure (whose leg distances are 4 and 2). Use the Pythagorean Theorem for Example 1.

Example 1:	Find the length of the left side of the parallelogram and state the reason it is this length.

<u>Looking Ahead 4.8</u>

Let us construct the diagonals of the parallelogram to see if we can learn something based on what we already know about them. We will call the parallelogram from above BOLD. We know that opposite sides are equal, so BD = OL and the distance of each is 4 units. The distance of BO and DL, which are equal, is $2\sqrt{5}$.

Example 2:	If angle OLD is 40° degrees, find the other three angles of the parallelogram. Is there anything we can say about consecutive angles of a parallelogram?



Draw in the diagonals. Do they appear to be equal? Measure them and see.
If you measure from the center to each vertex, you will see the diagonals do bisect each other.
Do the diagonals appear to bisect the angles? Measure them and see.

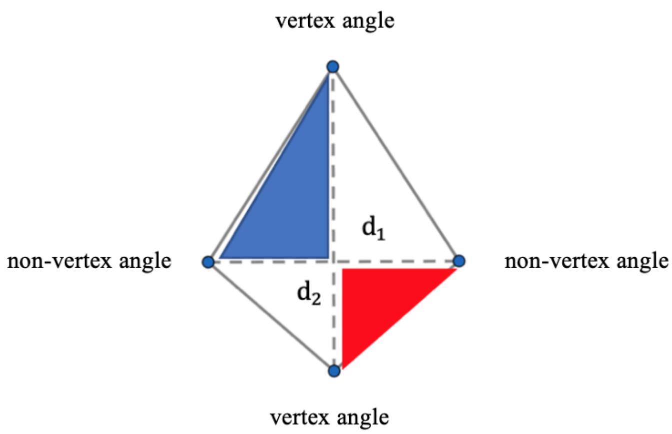
Section 4.9 Properties of a Kite

Looking Back 4.9

A kite, by definition, is a quadrilateral with two pairs of consecutive congruent sides.

Make a copy of the kite shown below. Fold along the longest diagonal, which extends from the top vertex to the bottom vertex. The top and bottom angles are called vertex angles. These are where the congruent sides meet. The angles on the right and left are called non-vertex angles. These are where the pairs of congruent sides meet.

Example 1: Fold along the long diagonal, connecting the vertex angles. Can you make a conjecture about the non-vertex angles?



Now, fold along the horizontal diagonal and you will see that vertex angles are not congruent.

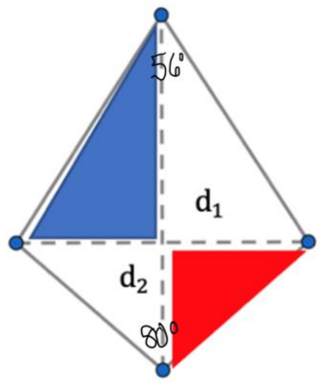
Example 2: What do you notice about the central angles of a kite? Measure the central angles and write a conjecture about the diagonals of the kite.

Example 3: Measure all four segments created by the kite. Can you write another conjecture about the diagonals?

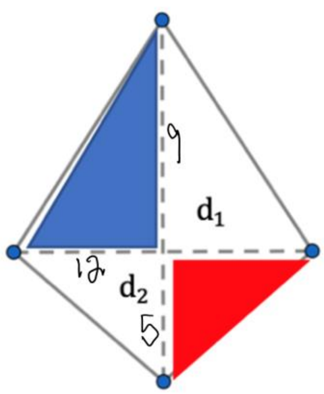
<u>Looking Ahead 4.9</u>

A kite is a quadrilateral with two pair of consecutive sides that are congruent, and two non-vertex angles that are congruent. The opposite sides of a kite are not congruent, and the vertex angles are not congruent. The diagonal that connects the vertex angles is the perpendicular bisector of the other diagonal and bisects the vertex angle.

Example 4:	Given the measure of the vertex angles, find the measure of the non-vertex angles.



Example 5:	Given the measure of the diagonal segments, find the perimeter of the kite.



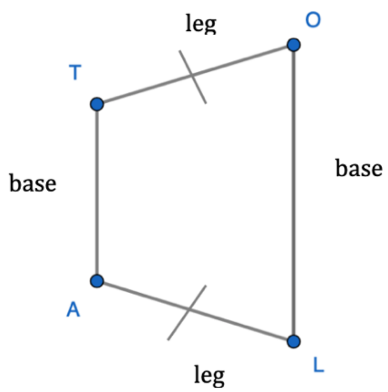
Like the parallelogram, the kite has two pairs of sides that are equal in length; however, they are not across from each other, but adjacent (next) to each other.

Section 4.10 Properties of Trapezoids

Looking Back 4.10

A trapezoid is a quadrilateral with one pair of opposite sides parallel. The parallel sides are called the "bases." The other pair of sides are called the "legs" and may or may not be parallel.

All the angles in a trapezoid have a sum of 360° because a trapezoid is a quadrilateral. If you copy the trapezoid below and cut out angle L and slide it up along line AL before stopping at angle A, you will see they form a straight angle of 180° degrees. If you slide angle L along line LO before stopping beside angle O, you will see there is a gap. Therefore, the consecutive angles between the bases of a trapezoid are supplementary.



AL is congruent to TO and LO is parallel to AT. ALOT is an isosceles trapezoid because the legs are equal.

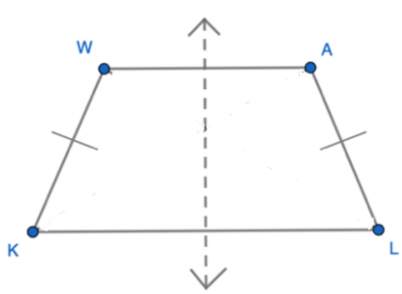
If the non-parallel sides of a trapezoid are congruent, then the trapezoid is called an isosceles trapezoid and has other special properties. We will investigate these properties in this section.

Looking Ahead 4.10

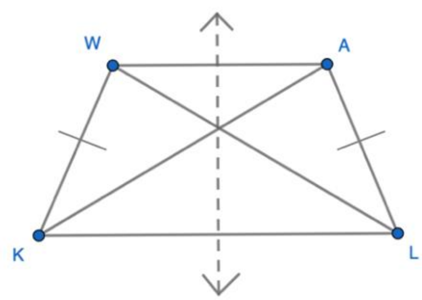
Let us tap into our prior knowledge of polygons again and see if we can use what we know about an isosceles triangle to learn new things about an isosceles trapezoid.

We know there are two congruent sides in an isosceles triangle; there are two congruent sides in an isosceles trapezoid as well. The base angles of an isosceles triangle are congruent to one another; do you think the base angles of an isosceles trapezoid are also congruent to one another?

Example 1: Measure angle K and angle L in the figure below to determine if the base angles of an isosceles trapezoid are congruent to one another.



Example 2: Put a point at the midpoint of WK and call it point E. Put a point at the midpoint of AL and call it D. Draw a line from E to D. This is called the median of the trapezoid. What do you notice about KE, EW, LD, and DA? (Note: The median line is sometimes called the “midline” or “midsegment”)

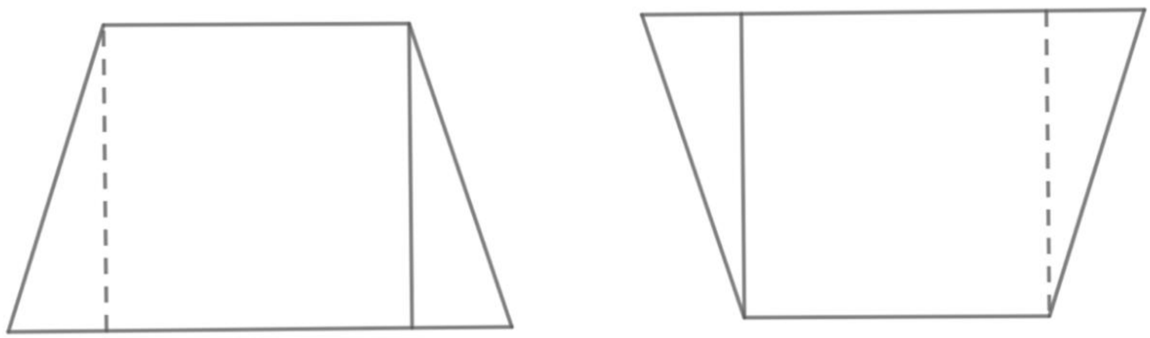


Example 3: The length of the median is the average of the length of the bases. If the length of the bases are 12 cm. and 22 cm., what is the length of the median? If the length of one base is 15 cm. and the length of the median is 11 cm., what is the length of the other base? How many lines of symmetry does an isosceles trapezoid have?

Example 4: What do you notice about the diagonals of the isosceles trapezoid in Example 2? Measure them and record the measurements.

The area of a parallelogram is $(b_1 + b_2)h$. Let us review the area of a trapezoid.

Example 5:	Find the area of the trapezoid with bases of lengths 7 m. and 11 m., and a height of 10 m.



In the United States, the trapezoid has at least one pair of opposite sides that are parallel. The trapezium has no pair of opposite sides that are parallel. In the United Kingdom, the trapezoid has no pair of opposite sides parallel. The trapezium has at least one pair of opposite sides parallel.

Section 4.11 Dilation, Vectors, and Translations

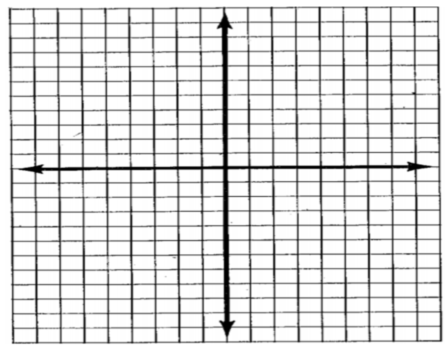
Looking Back 4.11

A figure or object is dilated when it is multiplied by a scale factor. An enlargement occurs when the points of a figure are multiplied by a scale factor greater than 1. A reduction occurs when the scale factor is between 0 and 1. In this type of transformation the angle measures stay the same after the dilation.

For reference, we have previously worked with scale factors when we studied matrices in Algebra 2.

Example 1: The matrix below represents points S(-1, -2), A(3, 0), and D(2, 4) on triangle SAD. Multiply the matrix by a scalar of 4 to quadruple the coordinates of the figure. Is this dilation an enlargement or reduction?

$$\begin{array}{c} S \\ A \\ D \end{array} \begin{bmatrix} -1 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}$$



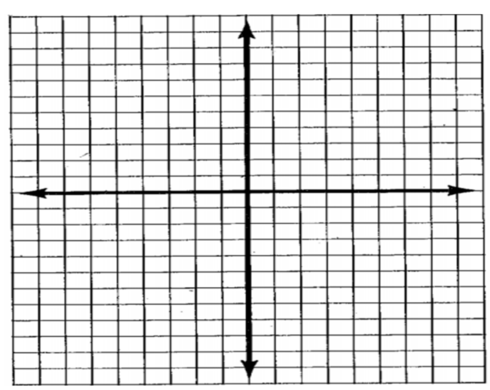
Using the coordinates for this dilation, the rule is $(x, y) \longrightarrow (4x, 4y)$. This is read, "x, y maps to 4x, 4y."

The triangle SAD is the pre-image and the triangle S'A'D' is the image.

S(-1, -2) ⟶ S′(-4, -8)
A(3, 0) ⟶ A′(12, 0)
D(2, 4) ⟶ D′(8, 16)

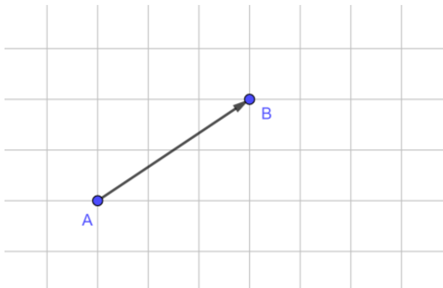
Example 2: Given the vertices of the pre-image, graph the trapezium (trapezoid in the UK) HOPE on the coordinate plane. Graph the image after a dilation by a scale factor of $\frac{1}{2}$. Is this an enlargement or reduction? What is the rule for the transformation using the coordinates below?

$H\ (-4, -1)$
$O\ (-2, -3)$
$P\ (3, -2)$
$E\ (1, 5)$



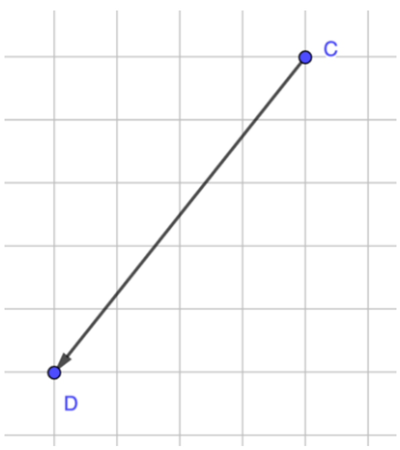
Looking Ahead 4.11

Rays can also be used as vectors. We will be using vectors in Pre-Calculus and Calculus to solve travel problems involving wind velocity. A vector has a head and a tail. The endpoint of a ray is the initial/starting point, the tail. The arrow of the ray is the terminal/end point, the head.



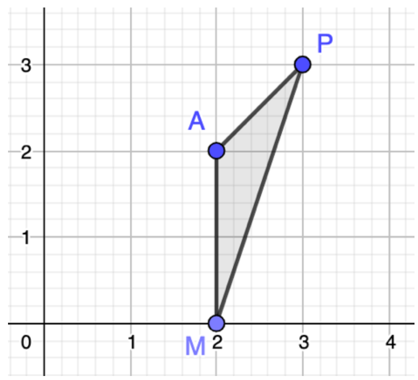
The vector AB is named $\overrightarrow{AB}$. The horizontal component is 3 units and the vertical component is 2 units. The component form is $\langle 3, 2 \rangle$ since moving from the initial point A to the terminal point B requires moving 3 units right and 2 units up.

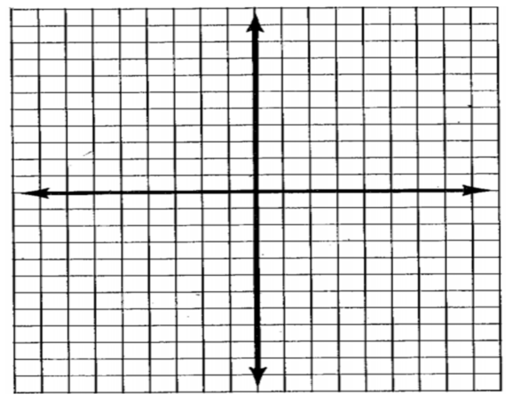
Example 3:	Write the vector $\overrightarrow{CD}$ in component form.



A translation is a transformation that moves every point of a polygon the same distance and the same direction. The initial polygon is called the pre-image (ΔMAP) and the mapping is the image ($\Delta M'A'P'$).

Example 4: Translate ΔMAP using vector $\langle 4, 2\rangle$.



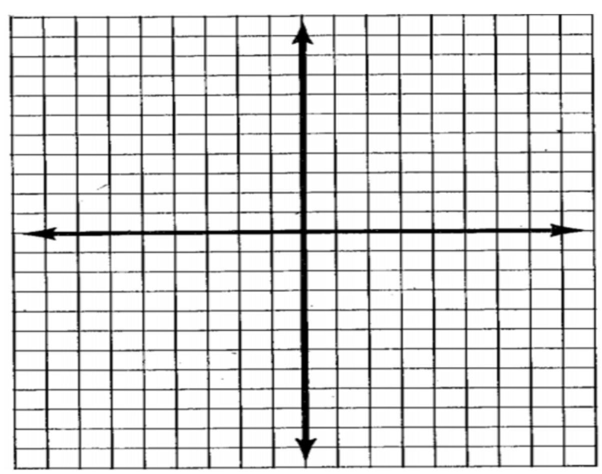


The rule for the translation using the coordinates of ΔMAP to ΔM′A′P′ is $(x, y) \to (x+4, y+2)$. This is read "(x, y) maps to $(x+4, y+2)$."

A translation is considered a rigid motion because length measures and angle measures remain the same after the transformation. A composition of transformations is like a composition of functions, where two or more transformations are combined in a single transformation. The composition of a translation remains a rigid or fixed motion.

Example 5: Given the rule $(x, y) \to (x-4, y-1)$, what is the translation using a vector in component form. If a quadrilateral FOUR has the following coordinates for the pre-image, what will be the coordinates of the image? Check the rigid motion using a graph.

$F\ (-2, 3)$
$O\ (-2, 6)$
$U\ (5, 3)$
$R\ (5, 8)$



Section 4.12 Reflections and Rotations

Looking Back 4.12

A similarity transformation allows the figure being transformed to keep the same shape but not necessarily the same size. Therefore, a dilation is a similarity transformation. A dilation combined with another rigid transformation, such as a translation, is a composition transformation. The image is similar to the pre-image. The angles stay congruent and the side lengths are proportional, so this composition of transformations is still a similarity transformation. The transformation maps one figure onto the other.

Two other rigid motions that are transformations, which keep both the side lengths and angle measures the same, are reflections and rotations.

First, let us investigate reflections. A line acts like a mirror and reflects the pre-image over or through the line of reflection. This can be the x-axis, the y-axis, the line $y = x$, or the line $y = -x$.

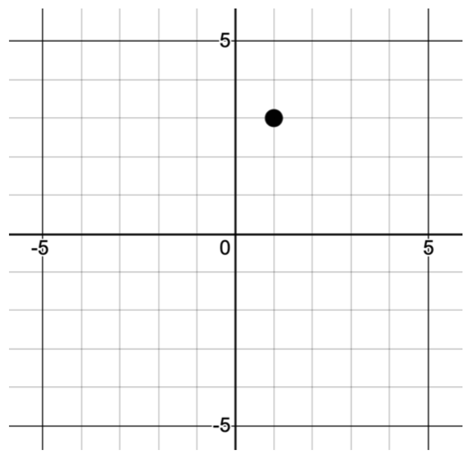
Example 1: The point $(1, 3)$ is drawn on the graph. Reflect it the four different ways and draw the images using colored pencils. Label each image with the corresponding letter. Explain the mapping of the (x, y) coordinates for each reflection. Use the Mira® to help you.

a) Reflection over the x-axis

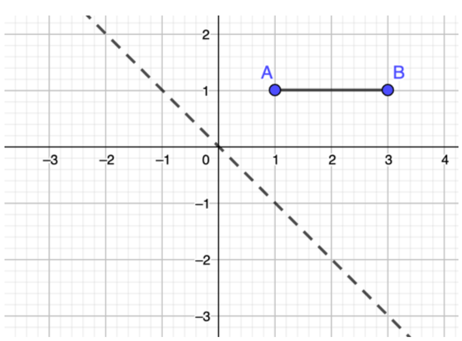
b) Reflection over the y-axis

c) Reflection over the line $y = x$

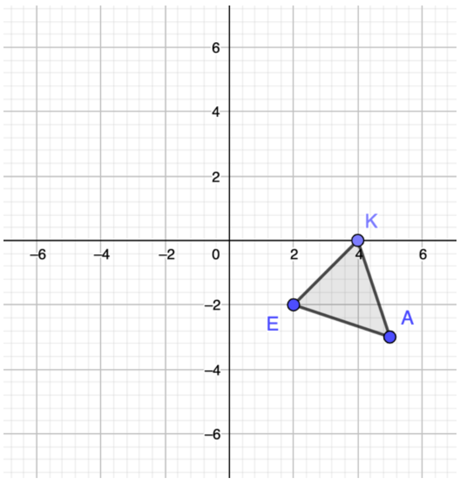
d) Reflection over the line $y = -x$



Example 2: Use the mapping rules from Example 1 to reflect the line segment AB over the line $y = -x$.



Example 3: A glide reflection is a composition of transformations involving a translation first, which is followed by a reflection. Apply the transformations to the pre-image of ΔKEA.



Translation: $(x, y) \rightarrow (x + 2, y + 1)$
Reflection over the line $y = x$

A rigid motion is also called an isometry. A transformation that changes the size and shape of a figure is a non-rigid motion.

Reflections are said to be "in the axis, over the axis, across the axis, or through the axis."

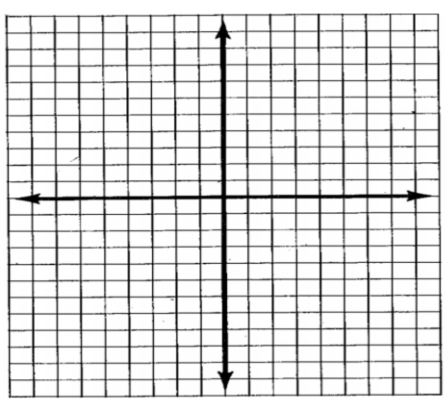
The coordinate rules for translations use h for the horizontal move and k for the vertical move such that $(x, y) \rightarrow (x + \mathrm{h}, y + \mathrm{k})$.

Looking Ahead 4.12

A rotation is another type of isometry or rigid motion. With a rotation, all the points of the original figure are turned an identical number of degrees the same direction from a fixed point, which is called the center. The rotation of degrees turned as well as the center point must be specified. The direction of rotation is always counterclockwise unless a clockwise rotation is specified.

In Algebra, you used tracing paper to draw rotations of figures. Dynamic geometry can also be used to demonstrate these rotations.

Example 4: Draw the triangle BEV using the coordinates B $(1, 3)$, E $(4, 3)$, and V $(3, 2)$. Let the origin be the center. Rotate the pre-image 90° about the origin. What are the coordinates of B′E′V′? What do you notice about the lengths of the sides, the measures of the angles, and the coordinates of the image?



In Example 4, the origin was the center of rotation. Rays drawn from the center of rotation to its pre-image point and to its image point form the angle of rotation.

Below are the other coordinate rules for rotations about the origin when a point (x, y) is rotated counterclockwise in increments of $90°$.

For a $90°$ rotation:
$(x, y) \rightarrow (-y, x)$

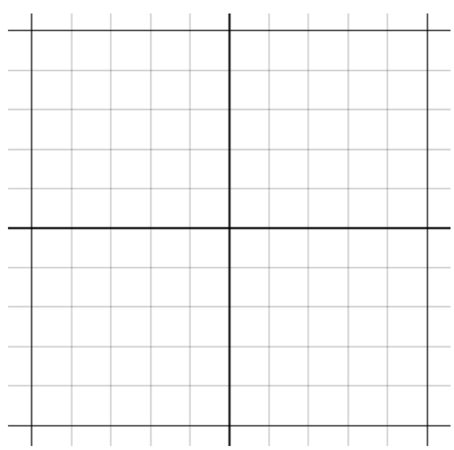
For a $180°$ rotation:
$(x, y) \rightarrow (-x, -y)$

For a $270°$ rotation:
$(x, y) \rightarrow (y, -x)$

Example 5: Use the coordinate rule for a $270°$ rotation for ΔBEV. Find the coordinates of the vertices of the image.

B $(1, 3)$
E $(4, 3)$
V $(3, 2)$

Graph the pre-image and image of the triangles.



Section 4.13 Pick's Theorem and Euler's Formula

Looking Back 4.13

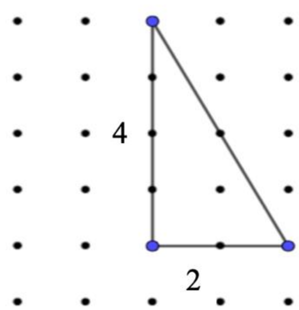
In this section, we will use the geoboard to discover a formula that Austrian mathematician George Alexander Pick discovered over one-hundred years ago.

George Alexander Pick was born in Vienna to Jewish parents and was educated at home by his father until he was 11 years old. After passing his college entrance exams, Pick studied at the University of Vienna in 1875. Only a year later, at age 17, he published a mathematics paper. Pick would go on to spend most of his career at the German University in Prague where he taught mathematics and physics.

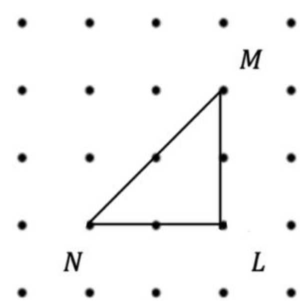
In 1900, Pick was appointed Dean of Philosophy and in 1910, he was instrumental in serving on a committee that appointed Albert Einstein to a chair of mathematical physics at the university. Pick and Einstein became close friends and shared an interest in music. Pick played in a quartet and introduced Einstein to the scientific and musical societies in Prague.

Sadly, after the German troops marched into Prague, they placed Pick and other famous Jewish people in a concentration camp in Theresienstadt. Two weeks after entering the camp, Pick died. He was 82 years old. Still, George Alexander Pick made many important contributions to the field of mathematics, including the geoboard, which we will be using in this section.

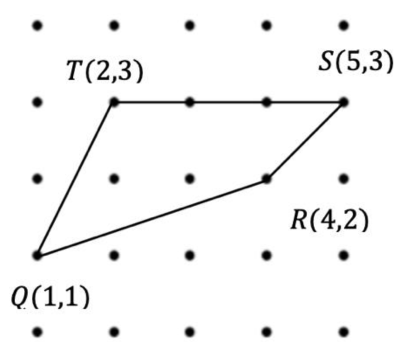
Pick's formula is used to find the area of the polygon on a square grid (the geoboard). His formula was published in 1899, but did not receive widespread acclaim until 1969, when it was seen again in Hugo Steinhaus' popular *Mathematical Snapshots*.



In the polygon to the left, there are 8 boundary points (lattice points touching the outside boundary of the triangle). There is one interior lattice point (it is not touched by the exterior of the polygon). The area of the figure is 4 square units.



In the polygon to the right, there are 6 boundary points, but no interior points. The area of the triangle is 2 square units.

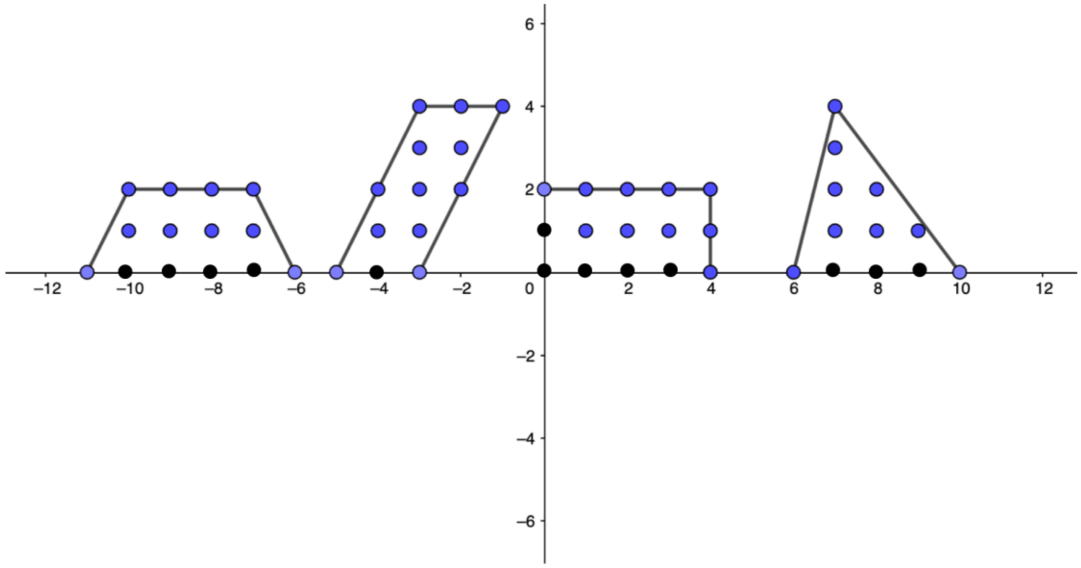


In this third polygon, the one to the left, there are 6 boundary points and 2 interior points. The area of the quadrilateral is 4 square units.

(In Pre-Algebra, you found all these areas using the chop strategy.)

Pick noticed a relationship between the boundary points, interior points, and area. Let us see if you can find the same relationship.

Looking Ahead 4.13



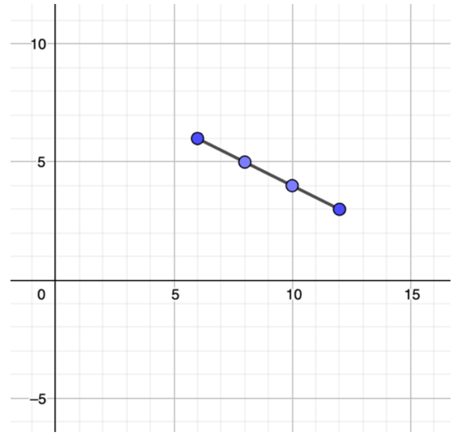
Each polygon above has an area of 8 square units. Count the boundary points (b) and the interior points (i) of each of the polygons and check them in the completed table below.

	Trapezoid	Parallelogram	Rectangle	Triangle
Boundary Points (b)	10	8	12	6
Interior Points (i)	4	5	3	6

Do you notice any relationship between the boundary and interior points and the area of each polygon? If not, let us do some more investigating.

If we graph the points (b, i) on the x-y coordinate plane and find an equation that fits the data, it may be easier to see the relationship.

The relationship between the boundary points and interior points is linear. The slope is $-\frac{1}{2}$ and the y-intercept is 9.



The relationship between the boundary points and interior points is linear. The slope is $-\frac{1}{2}$ and the y-intercept is 9.

In the graph above, the equation for the line is $i = 9 - \frac{1}{2}b$.

This equation could be rewritten as $\frac{1}{2}b + i = 9$. The area for all our polygons is 8, which is one less than 9, so the formula for our polygons would be $\frac{1}{2}b + i - 1 = 9 - 1$. Since this is for a specific area, the general equation for Pick's formula is as follows:

$$\frac{1}{2}b + i - 1 = A$$

Try this formula for the polygons above to see that it works.

In the practice problems, you will do some mathematical doodling to derive Euler's formula for polygons.