Geometry and Trigonometry Module 3 Foundations of Geometry

Section 3.1 Euclid's Postulates

Looking Back

In the previous courses, a connection has been made between algebra and geometry and how to solve problems has been demonstrated both algebraically and geometrically. The focus of these next four modules will be geometry, but connections to algebra will continue to be made.

Both algebra and geometry are mathematical, but geometry focuses more on the relationship between points, lines, planes, polygons, circles, surfaces (including area) and solids (including volume). The Greek word for "Geo" is earth and "metry" means measure. The literal translation of Geometry is "earth measure." Many Greek and Latin root words are used in geometry so that mathematics remains a universal language. To understand geometry better, we will begin with its origins.

Euclid, "The Father of Geometry," was a mathematician from Egypt in ancient times. He wrote thirteen books about geometry called The Elements, though we will limit our study to one volume. The basis for The Elements is five postulates. A postulate is a statement that is assumed to be true.

In his personal life, Euclid was the son of a merchant, a person who trades (buys and sells). Like the apostle Paul, Euclid traveled the populated parts of the world, though he was with his father. He compiled all that he learned on these travels about geometry into an organized series of books so that others could study them as well. So, much like the Bible, Euclid's volumes are a compilation of contributions from many sources. The introduction of The Elements presents the five postulates that lay the foundation for the rest of the series.

Euclid's five postulates:

1. A straight-line segment can be drawn joining any two points.

2. Any straight-line segment can be extended indefinitely in a straight line.

3. Given any straight-line segment, a circle can be drawn having the segment as the radius and one endpoint as the center.

4. All right angles are congruent.

5. If two lines are drawn that intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then it is inevitable that the two lines must intersect each other on that side if extended far enough. (This postulate is equivalent to what is known as the Parallel Postulate.)

We will learn more about these postulates and other theorems that can be derived from them as we move through geometry. We have done some informal proofs in algebra, but in this course, we will be doing formal geometry proofs, which follow a line of deductive reasoning and a series of steps to demonstrate that a theorem must be true. Remember, a conjecture is something that we surmise to be true most often because it appears to occur the same way over and over in our problem-solving. However, if one counterexample can be found then it disproves the conjecture. So, a conjecture is different from a formal proof. The postulates, like conjectures, are assumed to be true. Although a formal proof is not needed to prove a postulate, there must be no counterexamples to the premise proposed. Based on what we have learned we could say that formal proofs are to geometry as the Bible prophecies are to the prophets.

Let's look at the book of beginnings and see if we can write our own postulates. Here is a brief history of Genesis, which transliterated from Greek means "in the beginning:"

The stories of Genesis were passed down orally (by mouth) for years until God commissioned Moses to write the words down. Scribes continued this practice until the scripture was complete. Moses is the author of the first five books of the Bible, which Jesus confirms in the New Testament. These five books are called the Pentateuch (the Greek word for a five-volume book) or the Torah (the Hebrew word for "Law," often referring to the books of Moses: Genesis, Exodus, Leviticus, Numbers and Deuteronomy). Write down five possible postulates from Genesis.

For example:

1. In the beginning, God created the heavens and the earth.
2. God created the world in six days and rested on the seventh.
3. Only God can create something from nothing.
4. God created everything to bear fruit and reproduce after its own kind.
5. God created the first man and woman, Adam and Eve, who then populated the entire earth.

At the end of the day, theories and postulates are based on faith. Creationists believe the world was created by one true God. We did not observe creation taking place, nor can we reproduce it in a laboratory. We believe creation to be true because it is recorded in the Word of God. We conjecture that the Bible is evidence of the truth as the prophecies in it were fulfilled.

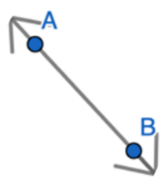
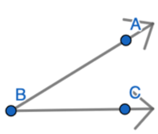
Some put their faith in the Theory of Evolution. The Theory of Evolution is called a theory because evolution has not been observed from its proposed origin and cannot be recreated in a laboratory.

<u>Section 3.2 Points and Lines</u>

<u>Looking Back 3.2</u>

This module is called "The Foundations of Geometry." We will continue exploring Euclid's postulates but in more detail. In this section, we will investigate points, rays, lines, and line segments. Throughout the rest of this module we will move through angles, measuring angles and angle relationships. In other modules, we will look at triangles, quadrilaterals and other polygons, circumference and area of circles as well as surface area and volume of a solid. We will be working in one, two, and three dimensions.

Let us get started by going over some basic geometric ideas and definitions. These definitions are understood to be true though point, line, and plane are considered undefined terms.

Description	**Figure**	**Symbol**	**Pronunciation**
<u>Point</u>: shows an exact location or position (has no dimension)	•	• Q	Point Q
<u>Line</u>: a set of points that extend in opposite directions without end (has one dimension- length but no width)		$\overleftrightarrow{AB}$ and $\overleftrightarrow{BA}$	Line AB
<u>Segment</u>: part of a line with two endpoints (often named in alphabetical order)		$\overline{CD}$ and $\overline{DC}$	Segment CD
<u>Ray</u>: part of a line with one endpoint (named in order starting with the endpoint)		$\overrightarrow{EF}$	Ray EF
<u>Angle</u>: two different rays that meet at the same endpoint		$\angle ABC$	Angle ABC

The following investigations will help you make connections between points, lines, and planes.

<u>Looking Ahead 3.2</u>

Investigation 1: Points are **collinear** if they lie on the same line. Tape a piece of string to the table and make sure to pull it tightly so the string is straight. Put three pennies on the string. Put a nickel on the table that is not on the string. Which coins are **collinear** (lie on the same line)? The pennies are **collinear**. The coins are **noncollinear** because the pennies do not lie on the same line as the nickel. If the nickel is moved so on the line with the pennies, the coins would be **collinear**. Remember, lines go indefinitely in opposite directions in the plane.

Investigation 2: When two points connecting a straight line lie entirely on a flat surface, that surface is a **plane**. Put a piece of blank paper on the table. Put a pencil on the paper so the whole pencil lies within the borders of the paper. The pencil represents a line and the paper represents a **plane**. Keep moving the pencil around on the paper but keep it within the borders. Imagine the paper extended in all directions to cover the table. It is still a flat surface. Imagine the pencil is extended in both directions to become the size of a ruler or even a yardstick, but it remains within the borders of the table. Now, the table represents the **plane** and the pencil symbolizing a ruler or yardstick represents the line.

Therefore, the formal definition of a **plane** is a set of points on a flat surface that extends without an end. It has two dimensions.

Investigation 3: Points are said to be **coplanar** if they all lie on one plane. Put three pennies on a piece of paper. The pennies are **coplanar**. Put the nickel on the table so it does not lie on the paper. Is the nickel **coplanar** along with the pennies? It is! Remember, the paper could be extended so that it is the size of the table it is lying on. Both represent the same flat surface, or geometrically speaking, the same plane. With the three pennies still on the paper (which is on the table), where could you place the nickel so it would not be **coplanar**? You could place it above or below the table somewhere on a shelf or on the floor. Even if you placed it beside the table to the right or left at the same level, it would be **coplanar** because the plane or flat surface of the table could be extended. When you hold the nickel above or below the table, the points are not on the same plane; therefore, they are **noncoplanar**.

Investigation 4: Lines intersect if they meet in one point. The definition of intersecting lines is “two lines with a point in common.” Fold a piece of paper in half top to bottom. Now, fold the same paper in half right to left. Put a penny on the point where the two lines meet. The penny should be in the center of the paper where the two lines cross. This is a point that is on the line going up and down and also lies on the line going across from left to right. Notice that the two lines meet to form what looks like a plus sign. These lines are called **perpendicular** if the two lines in each corner form what looks like the letter “L.” These “L’s” are called right angles and measure 90° degrees.

Investigation 5: Lines that lie on the same plane but do not intersect are parallel. The consecutive points on each line are equidistant from one another. Take a piece of paper and fold the left side over onto the right side. You now have a line that is in the middle of your paper. The line in the middle from top to bottom is parallel to both the right side and left side of the paper. The points on the right side of the paper are equidistant to the points on the left side of the paper. If you measure from the middle line to the right side using a ruler that is perpendicular to both lines you will see that they are the same distance apart all the way up and down the lines. If you take a piece of paper and fold it in half from top to bottom, you will have a line across the middle parallel to the top edge and bottom edge of the paper. This is perpendicular to the other fold.

The symbol for perpendicular lines is an upside-down T. The symbol for parallel lines is two straight lines vertical lines written side by side.

When you say that $\overleftrightarrow{AB}$ is **perpendicular** to $\overleftrightarrow{CD}$, geometrically, it is written as follows:

$$\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$$

When you say that $\overleftrightarrow{AB}$ is **parallel** to $\overleftrightarrow{CD}$, geometrically, it is written as follows:

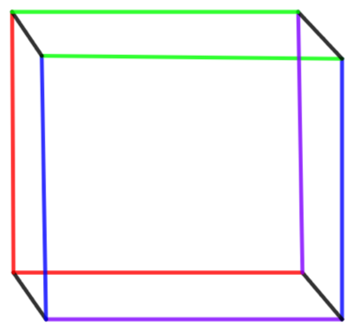
$$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$$

Investigation 6: **Parallel planes** do not intersect. Take a piece of paper and lay it on the table. Take another piece of paper and hold it up so it is parallel to the paper on the table. Where are two places in the room that you could tape the second piece of paper, so it is **parallel** to the paper on the table? You could tape it to the floor or the ceiling.

Investigation 7: Two planes **intersect** in a line. Look at the box in the figure below. The plane on the top intersects the plane on the front of the box and the plane on the back of the box (**green** lines). The two **green** lines are **parallel**. The **green** lines lie on the top plane (front and back). They also lie on separate planes, the top of the front plane and the top of the back plane. These planes are parallel. Each face of the box represents a plane. How many planes are there in a box? There are six faces so there are six planes. Locate the other parallel lines in the figure. *

Two adjacent planes meet at an edge. Count the edges. There are twelve. Lines that meet in a corner are **perpendicular**. We are assuming the corners of the box are square. The **red** lines are **perpendicular**. How many lines are perpendicular to the **purple** line on the bottom of the front face? There are two at each corner making a total of four. Locate the other perpendicular lines in the figure.

Let us take a closer look at the **purple** lines. Are they parallel? No, they are not going the same direction. All the points on each line are not equidistant to the points on the other line. Are they perpendicular? No, they do not intersect but lie on separate planes. These lines are called **skew** lines. They are not parallel, and they will never intersect one another. Lines that lie on different planes and go in different directions are called **skew**. Can you find other pairs of **skew** lines in the box?



* Parallel lines lie on the same plane but do not intersect. The points on one line are equidistant to the points on the other line. Two lines in a plane must intersect or be parallel to one another.

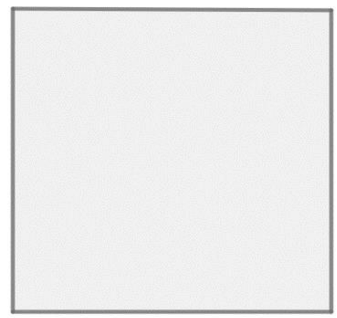
Skew lines have no intersections and are not parallel. Therefore, there must be three dimensions to have skew lines.

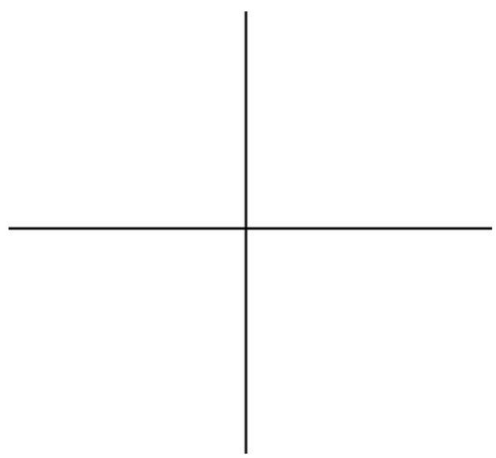
<u>Section 3.3 Line Design (Project)</u>

<u>Looking Back 3.3</u>

Now that we have learned a little bit about lines, we will integrate some art into our geometry by making line designs. Geometric designs are found throughout the natural world because God is an intelligent designer. Much geometry can be found in man-made art and in architecture that mimics God's designs. We will be performing several projects that are extensions of art in context throughout these geometry modules. The line design shapes will be investigated further and more mathematically when ellipses, hyperbolas, and conic sections are studied at the end of this course.

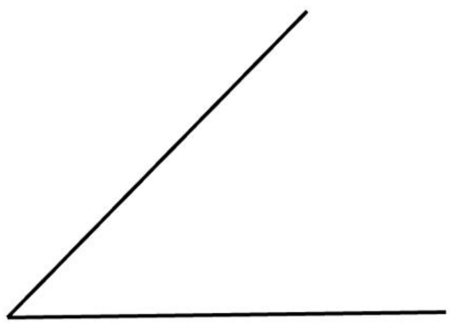
Using the three templates below, follow the demonstration of how to draw line design elements with colored pencils. All the designs we will be making in this section are some configuration of the following three designs.

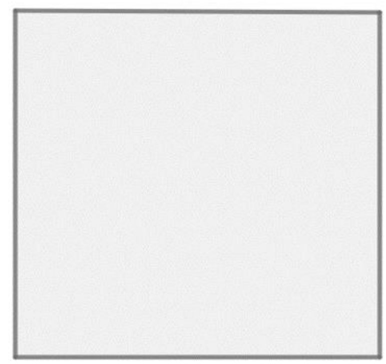




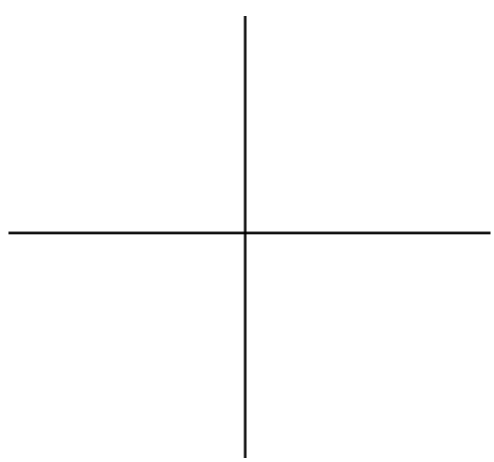
Looking Ahead 3.3

The following demonstration is created by drawing a parabolic curve inside the angle formed by two rays. The other design is created by inscribing the parabolic curve in the square.

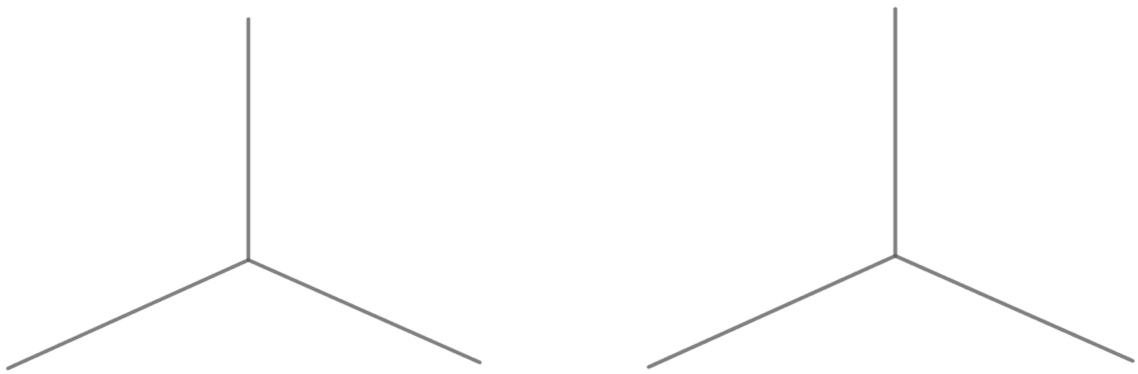




The design in the intersecting axis below becomes an astroid. An astroid is a hypocycloid curve with four cusps at a locus (center of focus) of a point on a circle that rolls inside a fixed circle with four times the radius. (We will learn more about loci at the end of this text.)



Now we will make a Deltoid Line Design. The deltoid is like the astroid, but the center is not comprised of four 90° angles but of three 120° angles.



The final demonstration will be an isosceles triangle. The base angles for the isosceles triangle create two design elements, which overlap, making it more complex than the previous designs.

When one parabolic curve overlaps the other, dramatic effects are created as you will see in the practice problems section. The project is for you to make your own creative line design.

Section 3.4 Tools of Geometry

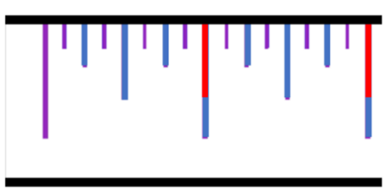
Looking Back 3.4

Since geometry means "earth measurement," it would stand to reason that when you study geometry, you will do a lot of measuring. Line segments, angles, and interior and exterior angles of polygons are just a few of the things measured in geometry. Tools are needed to do this. In ancient times, protractors (which you will use to measure angles) did not exist. Rather, people in ancient times would use a straightedge to draw straight lines. This straightedge would be an unmarked piece of wood with a straight side, but it did not include any specific measurements. Imagine a ruler or a yardstick with no markings on it. You could not use it to measure any distance, but you could trace along the side to draw a reasonably straight line. Perhaps you remember learning about how the Egyptians used knotted rope to mark off distances and make right angles in Pre-Algebra. It is quite remarkable that the great pyramids of Egypt were built without protractors!

A straightedge is used to draw straight lines through two points. The edge of a book could be used to do this and would be considered a straightedge. A ruler could also be used if the markings are ignored. The diagram below shows one inch of a ruler under magnification. Between 0 and 1 inch there are 16 marks. The measure from 0 to the first tick mark is $\frac{1}{16}$ (one-sixteenth) of an inch. The distances are shown on the ruler below in purple. The halfway point of the line is $\frac{8}{16}$, which simplifies to $\frac{1}{2}$. This makes sense! Since the whole line is 1 inch, half of the line would be $\frac{1}{2}$ inch.

Every second tick mark is shown in blue. There are eight of these tick marks in one inch since 16 parts divided into 2 tick mark increments is 8. Therefore, every tick mark shown in blue represents $\frac{1}{8}$. The halfway point is equivalent to $\frac{8}{16}$. Since they are equivalent fractions, they both can be simplified to $\frac{1}{2}$. If we start at 0 and count over 4 tick marks, we get to 4 out of 16 tick marks or $\frac{4}{16}$. That is the same as $\frac{1}{4}$ and there are 4 of those in one inch.

There are two red tick marks on the diagram. One is halfway between 0 and 1 and the other is at 1 because $\frac{2}{2} = \frac{4}{4} = \frac{8}{8} = \frac{16}{16} = 1$.



The U.S. Customary system (based on English units borrowed from the Roman and Anglo-Saxons) can be very confusing when trying to measure precisely. It is also used in Myanmar and Liberia. The metric system, which is used in all other industrialized countries , is the preferred standard unit of measure for mathematicians and scientists. It is much easier to do conversions in the metric system (such as centimeters to meters) because the base ten system is used. Doing conversions using the American system (such as inches to yards) is much tougher and more time-consuming. It is easier to make mistakes and the final conversions become less accurate.

In science, you will learn more about measuring with the metric system and will also better understand the difference between accuracy and precision. From the diagram above, it is easy to see that for something very small, using sixteenths to measure a distance is more precise than using fourths. Because the sixteenths are closer together, the answer is more exact, though it is still an approximation.

Although not in the diagram above, a normal ruler will show inches on one edge and centimeters on the other, which allows us to use the American system and/or the metric system. There are 10 millimeters in each centimeter. Each tick mark is one millimeter. Therefore, 15 tick marks represent 15 millimeters or 1.5 centimeters.

In one meterstick, there are 1,000 millimeters, 100 centimeters, and 10 decimeters.

Looking Ahead 3.4

Two other measuring devices that are used quite often in geometry are the compass and the protractor. The protractor is used to measure angles in degrees and will be explored more fully in Section 3.7.

A compass is used to draw circles with a given point as the center and a given length as the radius. The radius of a circle is a segment whose endpoints are the center of the circle and any point on the circle. Arc lengths can also be made with a compass to mark off a line segment. An arc length is a portion of the circumference of the circle, but not the entire distance around the circle.

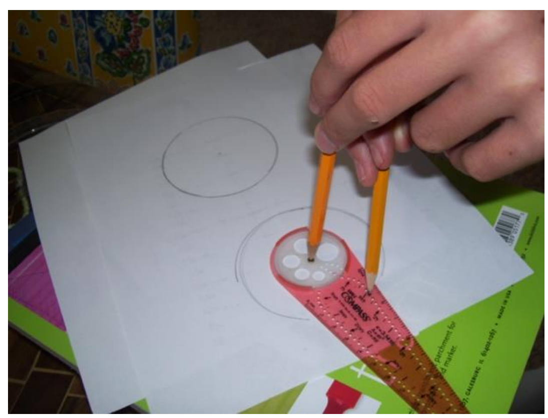
To use the compass, it is opened to a given distance that represents the radius of the circle at hand. The distance between the pencil and the edge with the metal tip is marked on the paper. The metal tip goes onto the paper and stays at that point, becoming the center of the circle. The pencil is turned completely around the paper to form the circle.

The compass used most often these days is plastic rather than metal and lies flat on a piece of paper. A pencil is placed in the center of the plastic circle and becomes the center of the circle that is being drawn. There are holes along each side that are marked 1," or $2\frac{1}{2}$," etc. if you choose to use the American standard of measure (if you choose to use the metric system, centimeters are labeled on the opposite edge). A second pencil is placed along a given measurement that will be the radius of the circle. The pencil in the circle stays at that point while the pencil making the radius is moved around the page in a circular motion to form the circle.

Moving back and forth can cause the lines to get a little unruly.

Below is a diagram showing the pencil moving forward and backward where the lines do not meet.

It is probably best to go in one direction around the circle.



You may prefer to use the metal compass and keep the point in place with one hand while turning the paper with your other hand.

You can make your own compass using a long piece of string with a piece of chalk tied to one end (this is how it was done long ago). Go outside to your driveway or a safe spot away from cars. Have a friend hold one end of the string down on the ground to make a point. This point will be the center of the circle. Tie a piece of chalk to the other end of the string and pull it tight so the string makes a straight line between you and your friend. Keeping the string tight and putting the chalk to the ground, move all the way around the person in the center of the circle. When you get back to where you started you should have completed a whole circle.

Example 1: Activity: Use a straightedge and a compass to construct a perpendicular line through $\overline{AB}$ (shown below). In the following instructions, the word “intercept” means intersect when referring to arcs.

1. Construct a line bisector of $\overline{AB}$. (To bisect means to cut in half so you will use a compass and straightedge to find the line that is in the middle of $\overline{AB}$ and is at a right angle to $\overline{AB}$.)

a) Open the compass to a length just over half of $\overline{AB}$ and draw an arc through the line from point A and another arc through the line from point B so they intercept both above and below the line.

b) Label the points of interception X and Y and connect the points in a straight line. Label the point of the intersection of the $\overline{XY}$ and $\overline{AB}$ as point O.

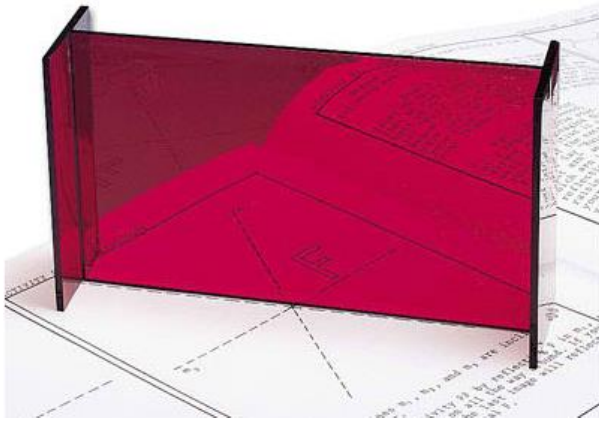
c) Measure from $\overline{XY}$ to A and from $\overline{XY}$ to B to make sure they are equal lengths.

Another tool used for geometric construction is made from plexi-glass and is called a Mira®. The Mira® is like a mirror but the image is reflected through it onto the other side. The Mira® has a beveled or indented edge, which is placed down so it touches the paper and faces you.

The beveled edge of the Mira® can be used to draw straight lines. It serves as an unmarked straightedge. Whatever is on the paper in front of the beveled edge can be seen on the other side as a reflection. Put the Mira® on a piece of white paper and put a pencil in front of it. Draw a line on the paper using the beveled edge of the Mira®. If you look through the Mira®, you will see a reflection of the pencil on the opposite side. Trace this reflection. Move the Mira® out of the line of view and you will see that the pencil is the same distance from the line as the drawing of the pencil’s reflection.

Draw a perpendicular line from the Mira® line to the tip of the pencil and measure it. Draw another perpendicular line from the Mira® line to the tip of the traced reflection of the pencil and measure it. They should be the same length. You could draw a straight line from the tip of the pencil to the tip of the pencil tracing and measure it and then divide it in two. Why does this give the same length?

Try drawing your name in front of the Mira® and trace its reflection on the paper.



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Draw a perpendicular line from the Mira® line to the tip of the pencil and measure it. Draw another perpendicular line from the Mira® line to the tip of the traced reflection of the pencil and measure it. They should be the same length. You could draw a straight line from the tip of the pencil to the tip of the pencil tracing and measure it and then divide it in two. Why does this give the same length?

Try drawing your name in front of the Mira® and trace its reflection on the paper.

Now, let us try some constructions using the Mira®.

Example 2:	Activity: Given $\overline{AB}$, draw a perpendicular line segment to $\overline{AB}$.

1. Put the Mira® on a paper and draw a line segment from the left side to the right side along the beveled edge (which is facing you). Label the leftmost point A and the rightmost point B.
2. Perpendicular lines look like crosses. Move the Mira® to point A so that you can see point B through it. Move the middle of the Mira® along the line segment until the reflection of part of the line segment and point A is seen on point B. Look through the Mira® and line up the reflection with the line segment on the paper. Draw a line segment along the beveled edge of the Mira® from left to right. Label the top point C and the bottom point D.

Section 3.5 Congruency

Looking Back 3.5

When you bisected a line segment in the previous section, you cut the line segment in half using only a compass. This is called a construction because you built the line segment bisector using one of the tools of geometry: a compass. If you wanted to check to see if you really did bisect the line segment, you would measure each half with the ruler to ensure they are the same length.

In the line segment below, point I is in the middle of points K and P; therefore, point I bisects line segment KP. Then the length of line segment KI is equal to the length of line segment IP. We can write this as follows KI = IP. This means the lengths are equal.

However, if point I were constructed as the bisector of $\overline{KP}$, then, geometrically speaking, $\overline{KI}$ is congruent to $\overline{IP}$. We use the word congruent to mean equal when we are talking about relationships between geometric figures. The geometric figures here are congruent. This means the segments are congruent.

The symbol for congruency is an equal sign with one wave of the approximately equal sign above it. It can be written as follows: $\overline{KI} \cong \overline{IP}$. This is read, "line segment KI is congruent to line segment IP."

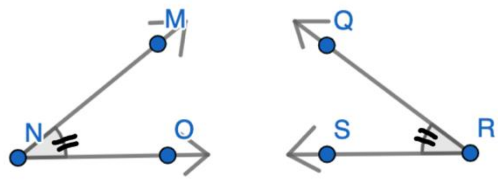
The Segment Addition Postulate states that "if a point (B) lies between two collinear points (A and C), then the sum of the shorter segments is equal to the longer segment."

$$AB + BC = AC$$

Example 1:	If $\overline{AB} \cong \overline{CD}$ and E is the midpoint of $\overline{AB}$, answer the following questions.

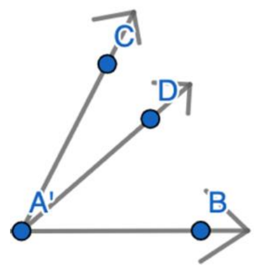
a) If $AE = 3x - 2$ and $CD = x + 11$, how long is AB?

Also, if the $m\ \angle N = m\ \angle R$, then $\angle N \cong \angle R$. The m in front of the angle means "measure of." Perhaps both angles have a measure of $45°$ degrees. Therefore, the measure of angle N is equal to the measure of angle R. Both angles are also congruent, which means they have the same size and shape. Therefore, angle N is congruent to angle R. Tick marks are used to show that the two angles are congruent.



$\angle MNO \cong \angle QRS$

The Angle Addition Postulate states that if a ray ($\overrightarrow{AD}$) is in the interior of an angle ($\angle CAB$), the sum of the measures of $\angle CAD$ and $\angle DAB$ is equal to the sum of the measure of $\angle CAB$.



$m\ \angle CAD + m\ \angle DAB = m\ \angle CAB$

Example 2: If $m\ \angle CAB = 87°$ and $m\ \angle DAB = 51°$ and $\overrightarrow{AB}$ is the ray adjacent to both angles, what is the measure of $\angle CAD$?

Example 3: If $m\ \angle CAD$ is 67° and the measure of $\angle DAB$ is 35° less than double the measure of $\angle CAB$, what is the measure of $\angle CAB$ and $\angle DAB$? The ray adjacent to the two angles is $\overrightarrow{AB}$. The ray lies on the interior of $\angle CAD$.

<u>Looking Ahead 3.5</u>

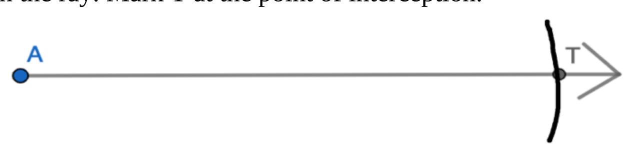
One type of construction that we have not done with our geometry tools is duplicating line segments and angles. If we make an exact duplicate, we have two figures of the same size and shape. The two line segments, the original and the duplicate, are congruent.

Let us try to duplicate $\overline{ON}$ so that we have $\overline{AT}$ with the same length as $\overline{ON}$.

How do you think you would do this? What do you need to draw a line? We need two points but let us start with one. Draw a point with a ray coming out of it that appears longer than $\overline{ON}$.

So, now we have a point and an extended ray. Mark the point on the left A. How can we use our compass to mark off the line segment on the ray that starts with A so that it is the same distance as ON?

Simply open your compass the distance of ON and put the point on A. Using the pencil, draw an arc the distance of t ON on the ray. Mark T at the point of interception.



There is no duplicate for you. You are unique. You were formed by God. Each person has a specific DNA that is unique to them. The world of science has not been able to duplicate this DNA to clone people. Even starting with your DNA, a clone would be impossible to create. Only God can create you and you are formed in His image just as He said in Genesis 1:26, "Let us make man in our own image."

Section 3.6 Constructions Using Technology

Looking Back 3.6

We have used the protractor and the Mira® to investigate parallel and perpendicular lines. In this section, we will use different technology to investigate parallel and perpendicular lines and construct a square.

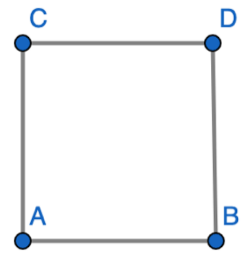
Constructions are different than drawings. If a square is hand drawn by using a geometer's tool to connect segments, it may look like all the sides are equal, but when measured, only the opposite sides are equal. This makes it a rectangle, not a square. Rectangles and squares and trapezoids and other quadrilaterals will investigated further in the next module.

$AB = 2.40$ cm.

$CD = 2.40$ cm.

$CA = 2.47$ cm.

$BD = 2.47$ cm.



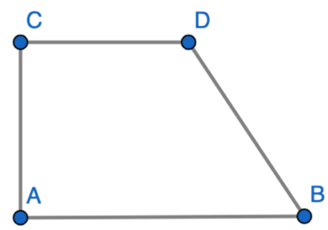
Moreover, when hand-drawing with a geometer's tool and attaching lines manually, if one point is moved, such as point B in the figure below, then the shape does not move as a whole, but one point moves and any line segments connected to it move as well. When point B is moved to the right, the square becomes what looks like a trapezoid. However, for the shape to be a trapezoid the lines on top and bottom would have to be parallel.

$AB = 4.02$ cm.

$DC = 2.40$ cm.

$CA = 2.47$ cm.

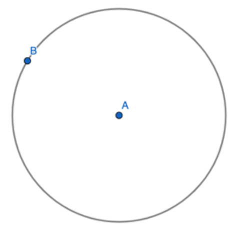
$BD = 2.95$ cm.



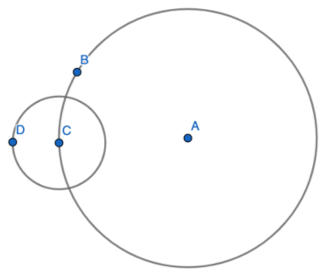
Looking Ahead 3.6

We will use a geometer tool for the remainder of this section to demonstrate how to construct a square. You can download GeoGebra™ to your computer for free. Or you can use the geometry page of the Ti-*n*spire® graphing calculator.

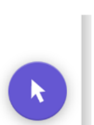
1. Click on the "Circle with Center" tool to draw a circle. Go to the blank canvas and click near the center. The first point becomes the center of the circle. Move outside this point and click again. The distance from the second point to the first point is the radius of the circle, which you should now see.



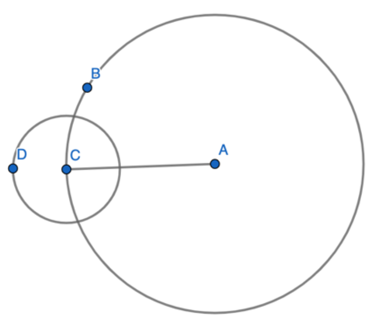
2. Keep "Circle with Center" selected. Now, go directly to the left of point A and click on the circumference of the circle. This will be the center of a new circle. Go directly to the left of this new center and click to make a circle that is smaller than the original circle.



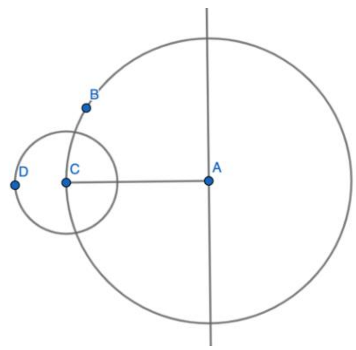
3. Click the purple icon with the white pointer to deselect "Circle with Center." Now, select "Segment." Click point A and then point C to create the line segment.



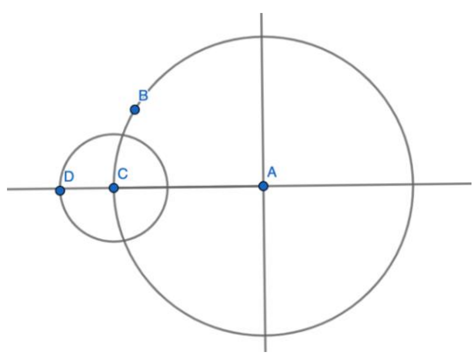
(Purple Icon)



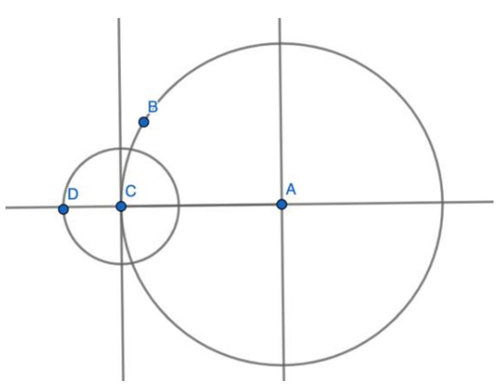
4. Click the purple icon to deselect "Segment." Click "More" and scroll down to select "Perpendicular Line." Click point A and then a point between A and C to construct a line that is perpendicular to $\overline{AC}$.



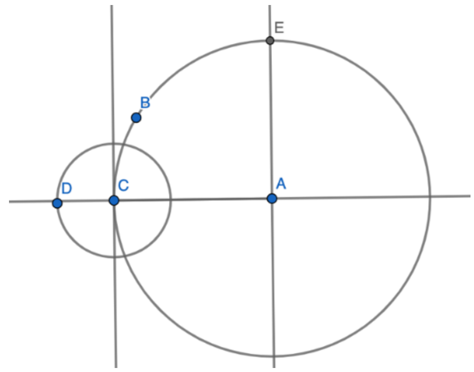
5. Keep "Perpendicular Line" selected. Click point A and then click on the line going straight down from it. This should create a perpendicular line to the one created in Step 4. This one is horizontal.



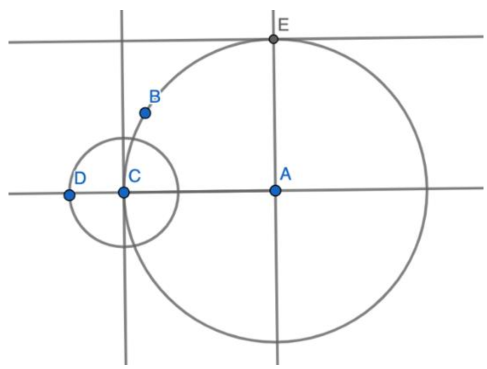
6. Keep "Perpendicular Line" selected. Click point C and then click on the horizontal line going through it. This should create the perpendicular line that goes vertical through C.



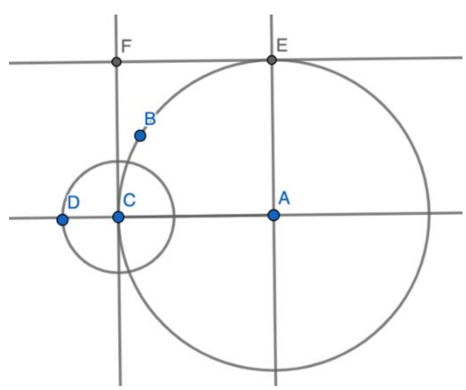
7. Click the purple icon to deselect "Perpendicular Line." Now, scroll up and click "Point." Click where the larger circle intersects with the vertical line going through point A at the top of the circle. Make this point E.



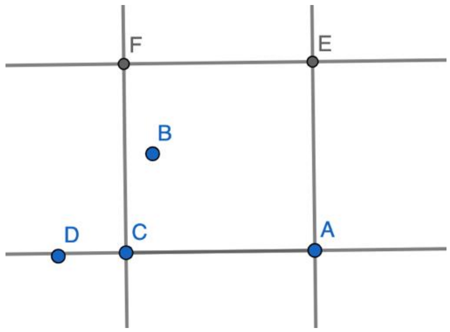
8. Click the purple icon to deselect "Point." Now, select "Perpendicular Line." Click point E and then click on the vertical line going down from E through point A. This should create a perpendicular line going horizontal through point E.



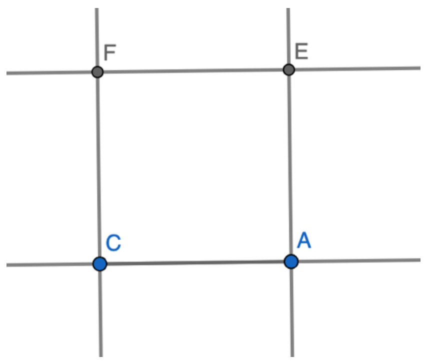
9. Click the purple icon to deselect "Perpendicular Line." Go up and select "Point." Click on the new intersection that has been made between the line going vertical from point C and the line going horizontal through point E. Make this point F.



10. The points F, E, A, and C are the corners of the constructed square. All we need is the square. Everything else can be hidden. Click the purple icon to deselect "Point." Now, select "Show/Hide Object," and click the circles. The circles should lighten. Then click the purple icon to deselect. The circles should disappear.



11. Now, click point B. Next, click "Show/Hide Object" and then "Show/Hide Label." Point B should disappear. Do this again with point D.



If you want to get rid of the lines extending past the points, you will have to hide the whole line, and then create new line segments between the four points F, E, A, and C.

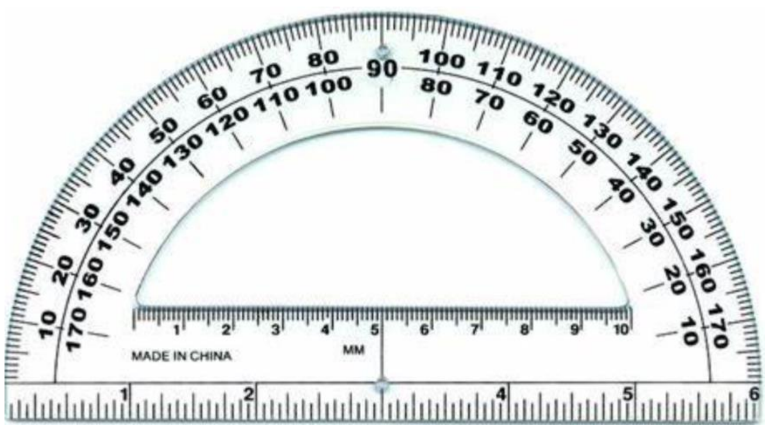
Section 3.7 Measuring and Drawing Angles

Looking Back 3.7

Another very important and useful tool in geometry is the protractor. The protractor is used to measure angles. Once measured, angles can be classified as acute, obtuse, or right.

A protractor is a semi-circle (half of a circle). The Egyptians thought there were 360 days in a year, and they thought the sun came up each day and the moon came out each night. Since the sun and moon are both circles, the Egyptians divided the circle into 360 equal units called degrees.

The symbol for degrees is " ° ." You should already be familiar with this symbol. If a circle is 360° then half of a circle is 180° (degrees). The bottom of the protractor may have a ruler along or near its base that measures in either inches or centimeters, or both inches and centimeters.



The numbers on the bottom of the arc of the circle increase from right to left. The degrees go from 0° to 180°. The largest number of degrees any angle can be measured on a protractor is 180°. This is called a straight angle. A straight line is a straight angle and measures 180°.

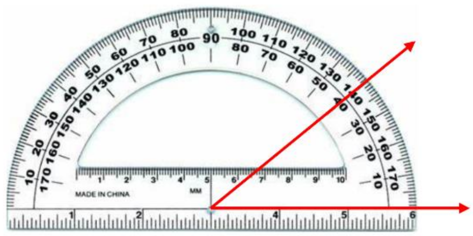
An angle that has its opening right to left will be measured using the numbers along the bottom of the arc, which increase from right to left. The bottom ray of the angle goes along the line on the base of the protractor. Be careful, it is not exactly on the bottom of the protractor, but a bit above it where the 0° starts. If you forget this, you will measure inaccurately.

It makes sense that the center of the angle should align with the center of the protractor. The vertex of the angle should be placed in the circle marked with crosshairs at the center of the protractor, which is directly below 90°. There is only one number there since it is halfway between 0° and 180°. This 90° angle is called a right angle and is at the same place whether you begin from the right or the left.

The use of protractors and precise measurements is a very practical application of mathematics in the "real-world." Construction engineers and architects use protractors for measuring. This means designers and builders use some of the same tools to complete their work. Of course, the architect uses a scale model for what is to be built and the builder must use this scale factor to build it. Still, in any case, both workers must be very skilled at mathematics to complete the job the best way possible.

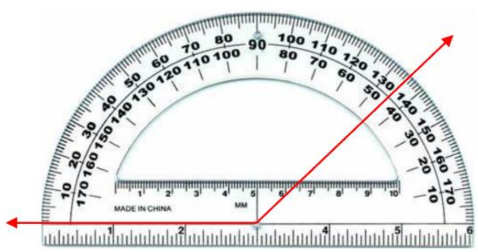
The sun provides the energy needed for organisms to survive. To complete its job, the sun must be in the right position. A few degrees off would make all the difference: if the sun were any closer or any farther, it would not provide the right amount of energy, warmth, and protection it does. It is safe to say that our Creator knew exactly where to put the sun in relation to the earth to provide us with a planet we can inhabit. This also gives us another real-world example of the importance of measuring accurately.

Example 1: Find the measure of the angle below.



The scale on the protractor is one tick mark is equal to one degree. Since there are 10° between each number, the long tick mark in the middle of two numbers is five degrees from the next number (higher or lower). When the bottom numbers increase, the top numbers decrease, so they are oriented in opposite directions. An angle that opens from left to right would increase from left to right. The numbers that increase from left to right are the upper numbers in the arc of the protractor.

Example 2: Find the measure of the angle below.



<u>Looking Ahead 3.7</u>

Drawing an angle is much like measuring an angle, but reverse thinking must be used. Instead of measuring the point where the upper ray crosses the protractor, we are marking the point where the ray should cross the protractor. Where are we marking it from? In other words, where is our starting point?

We need to draw the base ray first and align it, so the endpoint is at the center of the circle and the ray lies along the line at the bottom of the protractor. From there, the opening of the angle can be measured and marked, and the second ray can be drawn from that point to the endpoint of the first ray. This makes it the common endpoint (vertex).

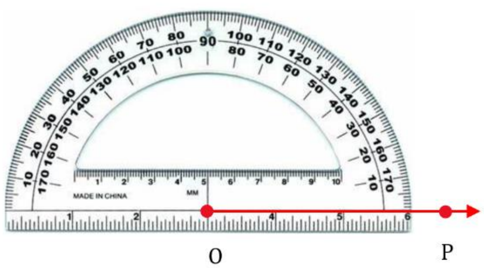
Now that you know how to measure angles, you can use tracing paper instead of a Mira® to make a kaleidoscope. In the module on polygons, mirrors will be used to make a kaleidoscope and later, when studying transformations, technology will again be used to design a kaleidoscope. Let us get started by drawing angles.

Example 3:	Draw $\angle NOP$ so that its measure is 33°.

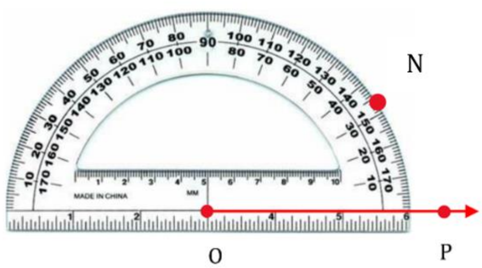
We know the vertex is the middle, so it is O since O is the common endpoint. The points at the extension of the rays are to the left and to the right of O. Therefore, the two rays are $\overrightarrow{ON}$ and $\overrightarrow{OP}$. Remember, both rays begin at O since that is the common endpoint.

Step 1: Draw $\overrightarrow{OP}$ as the base of the angle.

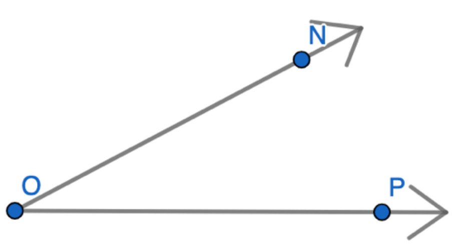
Step 2: Place the protractor on $\overrightarrow{OP}$ so that it is centered at O and the baseline is lined up along $\overrightarrow{OP}$.



Step 3: Put a point outside the protractor at 33°. We will use the bottom numbers because this point is less than 90°.



Step 4: Draw a line from N to O and put an arrow at N so that it represents the $\overrightarrow{ON}$, which is the other ray of the angle.



Do you think you can use your compass to duplicate that angle? Try It! Look back at your notes if you need help.

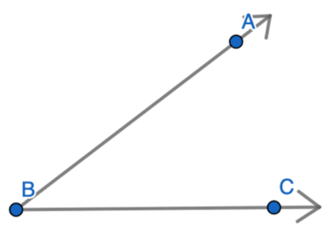
Section 3.8 Naming and Classifying Angles

Looking Back 3.8

Just as there are lines, rays, and line segments that are all parts of straight lines, there are also three types of angles. (In Module 6 when you study about triangles, you will learn that these angles have the same three names used for different types of triangles.)

Before we begin naming and classifying these angles, let us figure out what an angle is. The definition of an angle is an intersection of two rays that meet at a common endpoint called a vertex. You have seen this definition in a previous practice problems section. You have also learned the Angle Addition Postulate, which states that the measures of the two smaller angles that share an adjacent ray can be added together and the sum is equal to the measure of the larger angle.

Example 1:	Find the vertex of the angle and list two other ways to name the angle.



An angle can be named using only the vertex if there is only one angle at that point. Therefore, the first name for this angle is "$\angle B$."

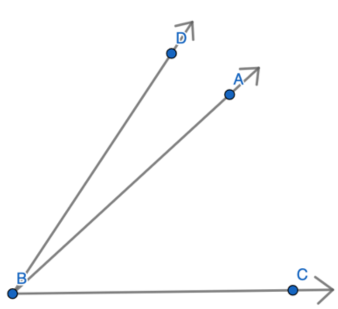
An angle may also be named from top to bottom by starting at the upper point on the upper ray of the angle and moving to the vertex and then out to the outer point at the base of the angle. Therefore, the second name for this angle is "$\angle ABC$."

The third name for this angle comes from the same method as "$\angle ABC$," but starts at the outer point on the base of the angle and ends at the upper point on the other ray of the angle. The third name is "$\angle CBA$."

Is "$\angle ACB$" another name for the angle?

This name does not work. Using your finger, if you start at a point on either ray and trace along the angle, you will see that the vertex is always in the middle. Therefore, ·B must be alone or in the middle when naming the angle with the three given points.

Example 2:	What is the vertex of the given angles? Name three different angles. How does the Angle Addition Postulate apply to this angle?



Why do you think you cannot use just the letter B when naming the angle?

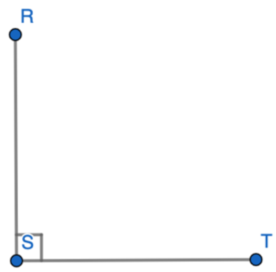
There is not just one angle that meets at that common vertex. There are actually three angles that meet at point B. There is the upper angle, the lower angle, and the larger angle that combines both. You will get a chance to name these angles in the practice problems section.

The three methods described previously use points to name angles. Calling the angle above "$\angle B$," "$\angle ABC$," or "$\angle CBA$" is much like calling an Elizabeth "Elizabeth," "Liz," or "Betty." It is like calling a female "woman," "girl," or "lady," or calling a male "man," "boy," or "guy." You can use different names to represent the same angle.

Let us move on to the types of angles.

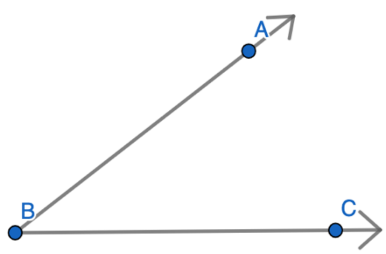
The first type of angle, which we have talked about before, is a right angle. You have been using right angles for a while, even in this module. Right angles are exactly $90°$. They often have a little square at the vertex to show that the lines are perpendicular and form a right angle. This way, you do not have to measure the angle to justify it. Below, $\angle RST$ is a right angle.

Example 3:	Justify the given right angle.



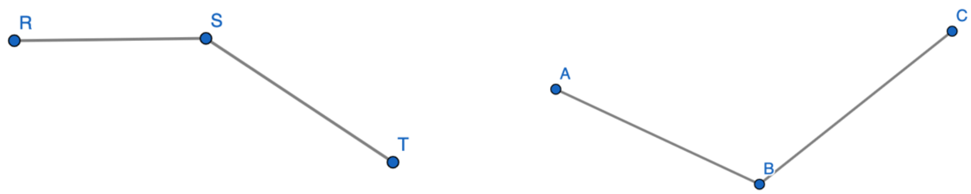
The second type of angle is an acute angle. An acute angle has a measure greater than $0°$ but less than $90°$. If you put both of your hands out palms down and put your fingers together and then make a movement with your thumb that mocks a duck quacking, you are forming an acute angle. That is "a-cute" little trick, isn't it? Your fingers move easily in this range of motion. Angle *ABC* in Example 4 below is an acute angle. It is smaller than the right angle above.

Example 4:	Justify the given acute angle. What is the vertex? What are three names for this angle?



The third and final type of angle is an obtuse angle. Obtuse angles have a measure greater than $90°$ but less than $180°$. If you put your hands out (palms down, fingers together), and then pull the thumb of your left hand straight to the right and until you are uncomfortable (while your other fingers stay lying flat and together and do not move), you are making an obtuse angle. The angle is the measure of space between the two rays that meet at the vertex so the orientation does not matter: the angle can be right side up or upside down. Both $\angle RST$ and $\angle ABC$ in Example 5 are obtuse angles.

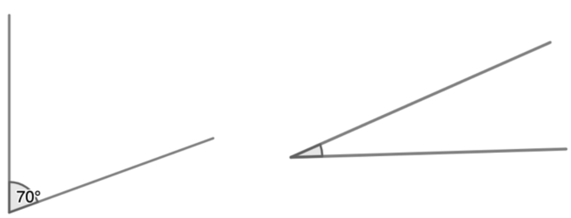
Example 5:	Justify the given obtuse angles and name each angle three different ways.



Before we conclude this section, we must discuss two more types of angles: complementary and supplementary angles.

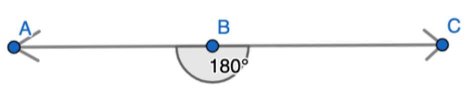
When two angles add up to 90° they are called complementary angles. Together they make a right angle. They complement each other, saying, “we are right together!” The two angles below are complementary. If the larger angle is 70°, what is the smaller angle?

Example 6:	The angles below are complementary. Find the measure of the unknown angle.



If the obtuse angle in Example 5 has a measure of 140°, it could be added to another angle with a measure of 40° for a sum of 180°. When two angles add up to 180° they are called supplementary angles. Supplementary angles can be next to each other and share a common side, but they can be separate as well. Angles that are next to each other are called “adjacent” (which means “next to”).

Example 7:	If two angles are supplementary and one of them is 20°, what is the measure of the other angle?

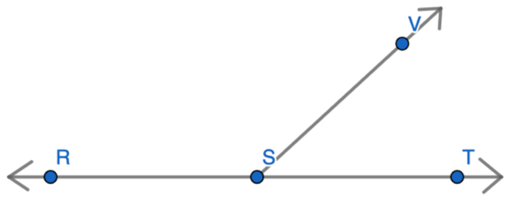


If $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays since point B lies between A and C on $\overleftrightarrow{AC}$, then a straight angle is formed. A straight angle is 180° and is made up of two opposite rays. Adding a ray in the interior of the angle with an endpoint of B forms two supplementary angles.

<u>Looking Ahead 3.8</u>

Now, we know that when an angle is fully open it is a straight angle with a measure of 180°.

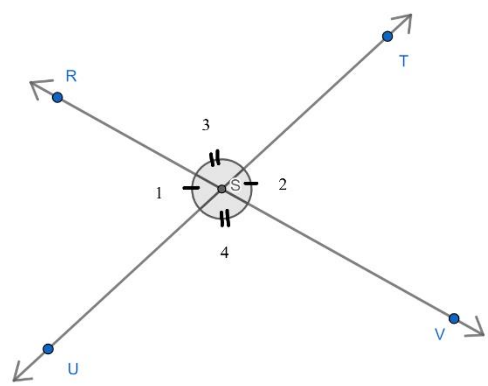
When two adjacent angles (angles with a common ray in the middle) have noncommon sides, which are opposite rays, they are called a linear pair. The angles in a linear pair lie next to each other on a straight line. This line forms a straight angle, so the angles are called a linear pair.



$\angle RSV$ and $\angle VST$ are a linear pair
$m\angle RSV + m\angle VST = 180°$

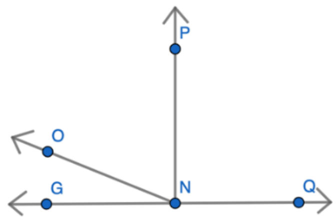
When two lines that are coplanar intersect at a point, they form angles between the sides that are opposite rays. These angles are called vertical angles. Vertical angles are always congruent.

Angle 1 and 2 are vertical: $\angle RSU \cong \angle TSV$. These are called vertical angles.



Example 8:	Name the other pairs of vertical angles in the diagram above.

Example 9: Examine the diagram below and list all the conclusive truths given that $\angle GNQ$ is a straight angle and $\angle PNQ$ is a right angle.



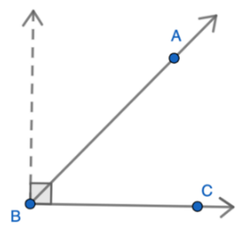
Example 10: Prove that $\angle GNQ$ in the diagram above is a right angle.

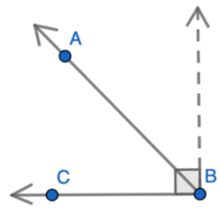
Section 3.9 Angle Relationships

Looking Back 3.9

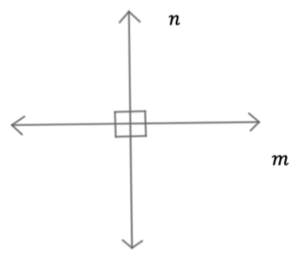
We have learned that angles can be classified as right if they are 90° (and form a square angle), obtuse if they are greater than 90° but less than 180°, and acute if they are less than 90°, but greater than 0°.

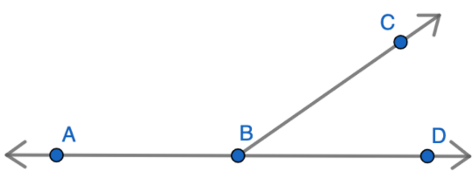
When two rays meet at a common endpoint, they form an angle. That common endpoint is the vertex of an angle. The angle below can be called $\angle B$, $\angle ABC$, or $\angle CBA$. Point B is the vertex so it is found between the points on each ray and stands alone.





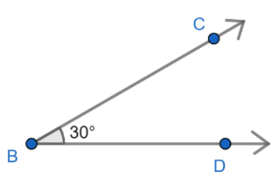
The angle formed in the diagram above is acute because it is smaller than a right angle. The dashed line shows where the right angle would be.
The sides of a right angle are perpendicular. Perpendicular lines meet at a point that forms an “L” shape, a “T” shape, or a “+” shape.

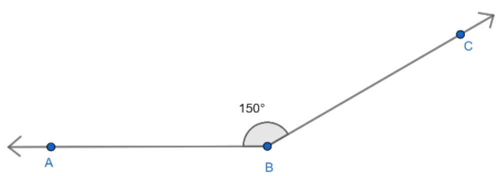




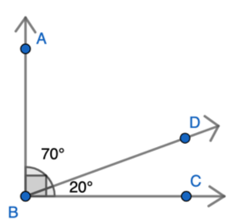
A straight line has an angle of 180°. This is the angle of a semi-circle, which is the shape of a protractor. If you draw a ray coming out of the straight line or straight angle, you get two angles that are adjacent (next to each other) because they share a common side (the ray) and a common vertex.

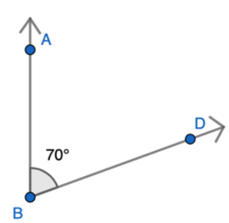
The adjacent angles, $\angle ABC$ and $\angle DBC$, are supplementary because they add up to 180°. Even if they are separated, they are still supplementary.

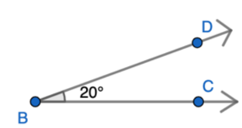




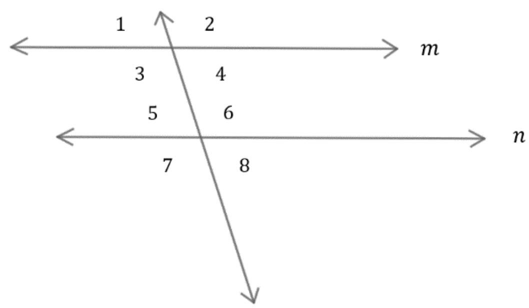
If you put a ray in between the two rays on the side of a right angle, you once again get two angles. These angles are adjacent (side by side and connected by a common ray and common vertex). They are called complementary angles because they add up to 90°. Again, even if they are separated, they are still complementary



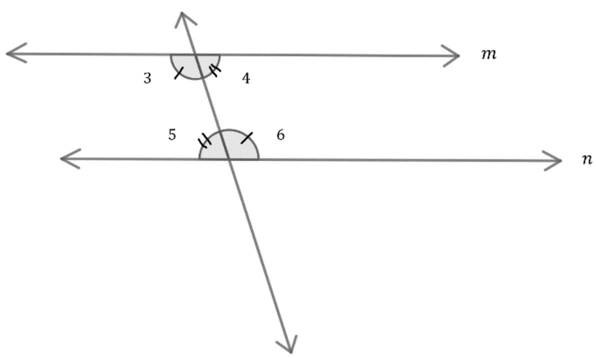




Example 1: A transversal is a line that intersects two or more coplanar lines at different points. The transversal below cuts through two parallel lines, m and n. All the angles are numbered. Find all the pairs that are supplementary. Two angles with corresponding positions are called corresponding angles. Find all the pairs that are corresponding.



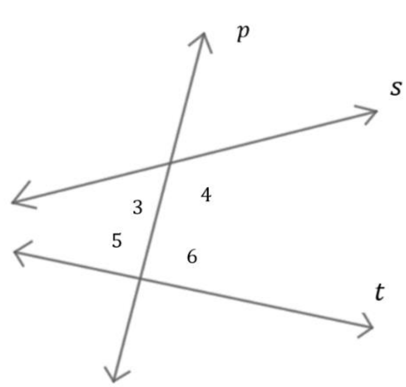
There are other special angles as well. On the diagram, ∠4 and ∠5 are an example of alternate interior angles. They are called "alternate" because they are on opposite sides of the transversal and "interior" because they are on the inside of (or between) the two parallel lines. Can you find the other pairs of alternate interior angles?



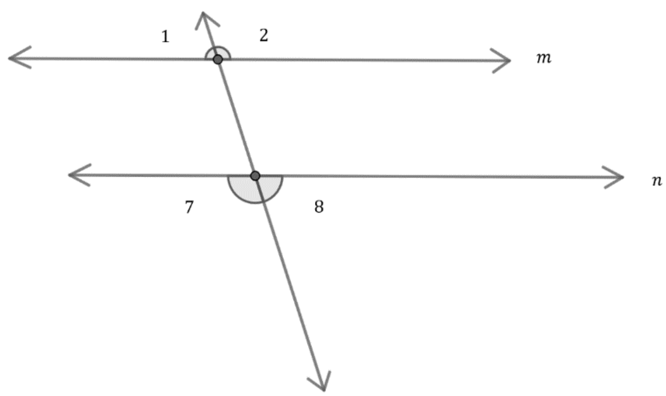
The two angles on opposite sides of the transversal, ∠3 and ∠6, are also alternate interior angles. When two parallel lines are cut by a transversal, four angles are formed between the parallel lines, which are two pairs of alternate interior angles.

Trace ∠3 on tracing paper and then rotate it around and slide it until it fits over ∠6 so you can see that they are equal. Do the same with ∠4 and ∠5. Two of the angles are acute and two are obtuse. Can you name them?

When a transversal intersects two other lines at different points, the lines do not have to be parallel. In the diagram below ∠3 and ∠6 as well as ∠4 and ∠5 are still alternate interior angles. However, they are not congruent. When the lines cut by the transversal are parallel, then the alternate interior angles are congruent (we will prove this in the last section of this module).



Example 2: Doesn't it make sense that if we have alternate interior angles, we will also have alternate exterior angles? Using logical deduction, name them in the figure below. (Again, alternate means they are on opposite sides of the transversal and exterior means they are outside the parallel lines, not between them.) Find the two pairs of angles and use tracing paper to demonstrate they are equal.

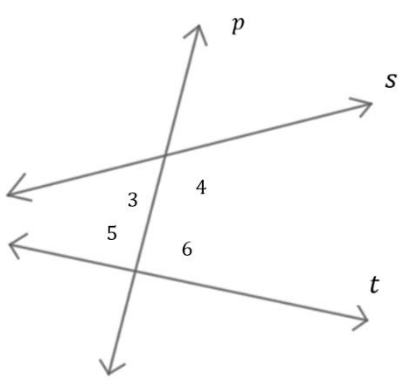


Did you notice that the alternate interior angles are congruent, and that the alternate exterior angles are congruent? The geometry found in God's natural world is orderly and pleasing to the eye. Though this may not surprise you, it will continually inspire awe in people who observe it.

The pair of angles in Example 1, $\angle 3$ and $\angle 5$, are called consecutive interior angles, and they are supplementary when the lines are parallel. Trace $\angle 3$ and rotate it next to $\angle 5$ in Example 1 and you will see they form a straight angle equal to $180°$.

Can you name another pair of consecutive interior angles? They are on the same side of the transversal and lie between the two parallel lines.

Example 3: Name two pairs of consecutive interior angles in the diagram below. Consecutive means on the same side of the transversal and interior means in between the two lines cut by the transversal. Are they congruent? If the lines *s* and *t* were parallel, what could you deduce about the consecutive angles? (See Example 1).



Section 3.10 Theorems of Parallel and Perpendicular Lines

Looking Back 3.10

Axioms and postulates are truths that are accepted without proof. The Lord says that we are blessed if we believe He exists without seeing Him. This is called faith: believing without seeing. We accept the truth of His existence based on His creation, which surrounds us.

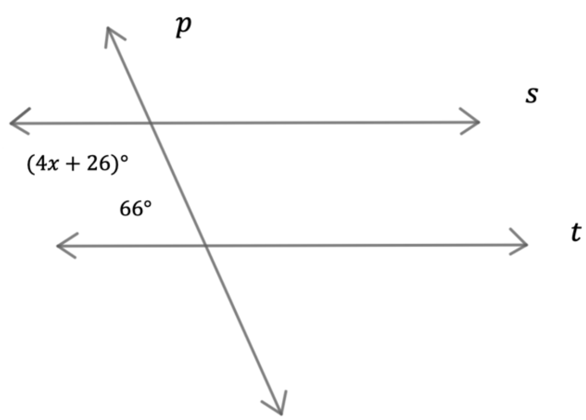
When statements can be proven to be true, then they are called theorems.

The Alternate Interior Angles Converse Theorem states that if two lines are cut by a transversal so that alternate interior angles are congruent, then the lines are parallel.

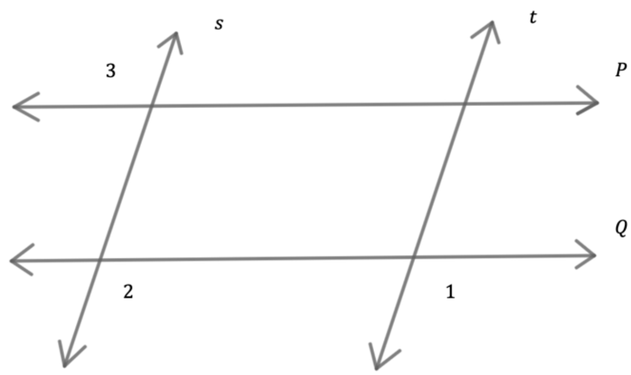
The Alternate Exterior Angles Converse Theorem is very similar. It states that if the alternate exterior angles of two lines cut by a transversal are congruent, then the two lines are parallel.

Example 1: Write the converse of the Consecutive Interior Angles Theorem, which states that if two parallel lines are cut by a transversal, then the pairs of consecutive interior angles are supplementary.

Example 2: For the diagram below, find the value of x that makes line s parallel to line t by the Consecutive Interior Angles Converse Theorem.



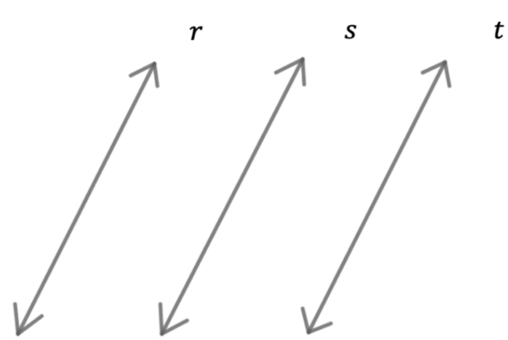
Example 3: In the figure below, let line s be parallel to line t. Given that $\angle 1 \cong \angle 3$, prove that line P is parallel to line Q. To do this, use the Transitive Property of Congruence, which states that if two angles are congruent to the same angle, then the angles are congruent to each other.



Looking Ahead 3.10

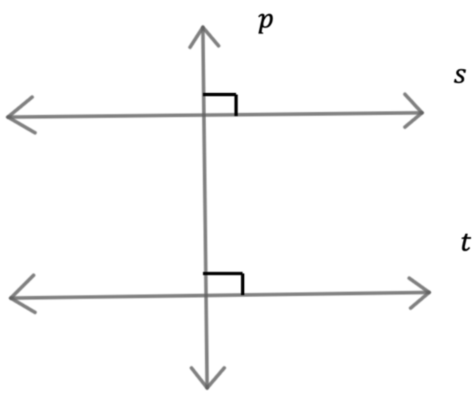
There are a few more theorems to present concerning parallel and perpendicular lines.

The Transitive Property of Parallel Lines states that if two lines are parallel to the same line, then they are parallel to each other.



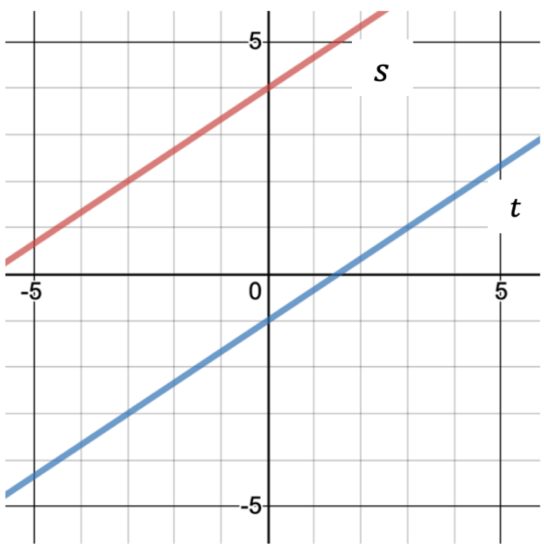
We will investigate the Perpendicular Transversal Theorem in the practice problems for this section.

Example 4: The Lines Perpendicular to a Transversal Theorem states that if two lines in a plane are perpendicular to the same ____________________, then the two lines are ____________________ to one another.



If $s \perp p$ and $t \perp p$, then s______t.

Example 5: The Slopes of Parallel Lines Theorem states that any two distinct non-vertical lines in a coordinate plane are parallel if and only if they have the same ____________________.



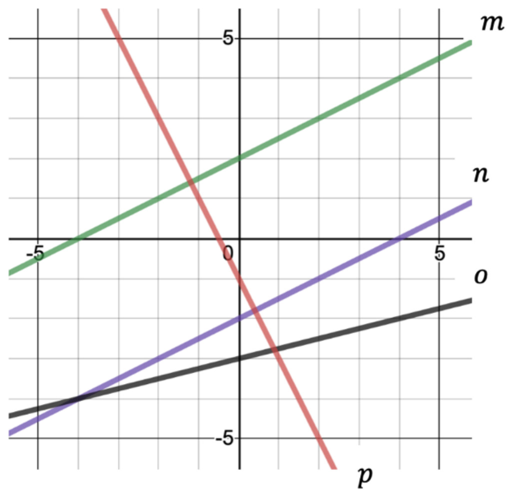
If $m_1 = m_2$, then s_______t.

Find the slope of s and the slope of t. Are the lines parallel?

Example 6: The Slopes of Perpendicular Lines Theorem states that any two non-vertical lines in a coordinate plane are perpendicular if and only if the product of their slopes is -1.

If $m_1 \cdot m_2 = -1$, then the lines are perpendicular.

In the diagram below, tell which lines are parallel and perpendicular.



Section 3.11 Inductive and Deductive Reasoning

Looking Back 3.11

In previous examples and problems, you were able to justify right, acute, and obtuse angles by applying the definitions in a visual context. Each of these justifications is a conjecture, which is an unproven statement based on observations and patterns. This is a form of inductive reasoning. Inductive reasoning is a method of reasoning in which general conjectures are formed from specific details.

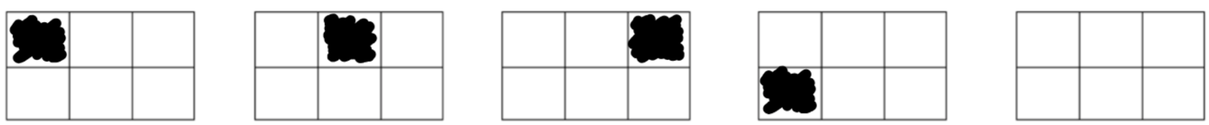
Looking Ahead 3.11

Inductive reasoning works from specific to general. Here is a simple example: Every day I leave for work at the same time. Every day I am on time for work. The conclusion is that if I leave for work at the same time each day, then I will arrive on time.

However, while this reasoning may be true most of the time, it does not account for obstacles that may be encountered on my drive to work. There could be traffic, snow, or any number of things that could slow me down.

Let us go through some examples where inductive reasoning is used.

Example 1: Completing a series is based on conjecture and inductive reasoning. Complete the fifth sketch in the series below using inductive reasoning.



Example 2: In the sequence below, what will be the next addends and sum?

$$1 + 1 = 2$$
$$1 + 2 = 3$$
$$2 + 3 = 5$$
$$3 + 5 = 8$$

Just one counterexample proves a conjecture to be false. For a conjecture to be true, it must be true in all cases.

Deductive reasoning works from general to specific. It can be a series of logical statements. For example, I John 4:7 says, “Beloved, let us love one another, for love is from God; and everyone who loves is born of God and knows God.” In this case, God’s love is the general example and “everyone who loves” is the specific example. The logical argument that follows this statement is that if I love others, then I am born of God and I know God.

I John 4:8 says, “The one who does not love, does not know God, for God is love.” The logical argument that follows this statement is that if I do not love others, then I do not know God.

Those are large-scale examples of deductive reasoning. Here is a simpler example: I leave the house every morning at 8:00. It takes me 30 minutes to drive to work. The logical argument that follows is that I arrive at work each day at 8:30.

Now, this may not always be true. I may be slowed down by an accident or construction on my way to work. However, the conclusion is inferred from the argument. Deductive reasoning starts with a hypothesis and then, after observation, experimentation, and data-gathering, reaches a conclusion.

Here are a few more examples of deductive reasoning:

1. The maple is a tree and all trees have bark, so all maples have bark.
2. People have cells in their bodies and all cells have DNA, so people have DNA in their bodies.
3. All birds have feathers and cardinals are birds, so all cardinals have feathers.

The premise is called P (hypothesis) and the argument is called Q (conclusion). If P, then Q so Q given P.

In geometry, the following would be an example of this:

- All triangles have three sides
- A figure has three sides
- The figure is a triangle

In the following example, notice that deduction is not used because the conclusion does not logically follow the argument:

- All odd numbers are integers
- All even numbers are integers
- Therefore, all odd numbers are even numbers

This is not true because odd numbers and even numbers are different sets of numbers and do not share any members in common.

Here are some examples of deductive reasoning in which the conclusion is not necessarily true:

- All observed basketball players are tall; all basketball players must be tall
- Janet is a math teacher; all math teachers are smart; Janet is smart
- Takiya is a ballerina; all ballerinas are tall and thin; Takiya is tall and thin

Now, after looking into both deductive and inductive reasoning, you should be able to tell that deductive reasoning is more specific and inductive reasoning is more general, which makes the conclusions for inductive reasoning broad and open-ended.

There is a kind of reasoning that stems from deductive reasoning called a syllogism. A syllogism is a logical argument based on two or more statements that are assumed to be true. Here is how a syllogism works:

- The first two statements are called a premise and provide clues to support or prove the third statement, which is called a conclusion.
- The argument begins with the words “all,” “no,” or “some.” The argument uses the verbs “is,” “are,” and “are not.”
- The conclusion, or third statement, begins with “therefore.”
- The premise may not make sense, but the important part is that it supports the conclusion.
- If the conclusion is logically based on the premise, then the syllogism is VALID.
- If the conclusion is not logically based on the premise, then the syllogism is INVALID.

Follow the steps below for the example to see how to test a syllogism with Venn diagrams. Tell whether the conclusion is VALID or INVALID.

Example 3:	All trees are green; all pines are trees; therefore, all pines are green.

Step 1: Draw three overlapping circles. Label each section with a different number from 1-7 going from left to right, top to bottom.

Step 2: Find which circle represents each category. Label the categories. The three categories are trees, green, and pines.

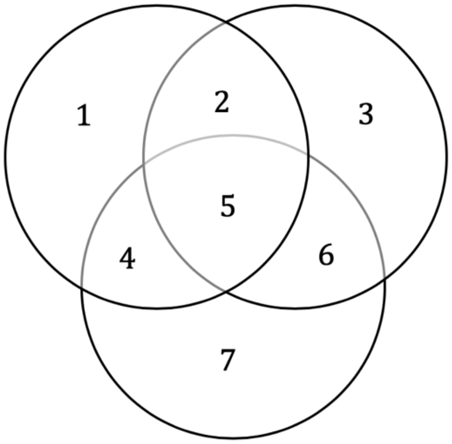
Step 3: The first premise, "All trees are green," means there are no trees that are not green. Shade in the part of the circle where trees do not overlap green. This means this set is empty.

Step 4: The second premise, "all pines are trees," means there are no pines that are not trees. Shade in the entire pines circle that is outside the tree circle.

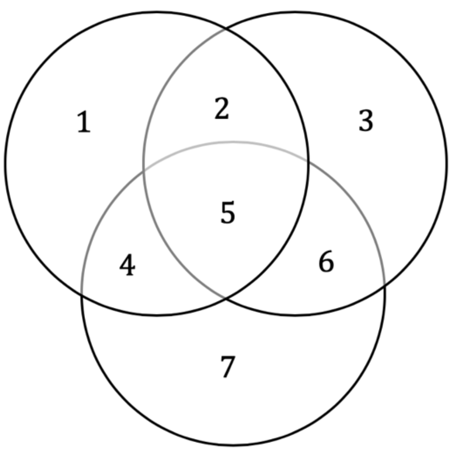
Step 5: The conclusion states "all pines are green." Look at the diagram. The shaded part of the pines circle means there are no pines there. All pines are in the unshaded part, which is labeled "5." Check to see if all the unshaded parts of the pine are also unshaded parts of the green. If they are, the conclusion is VALID. If they are not, the conclusion is INVALID.

Test the following syllogisms using Venn diagrams. State whether the conclusion is VALID or INVALID.

Example 4:	All cats are fish-eaters; all fish-eaters are animals; therefore, all cats are animals.



Example 5:	All houses are big; all cars are big; therefore, all houses are cars.

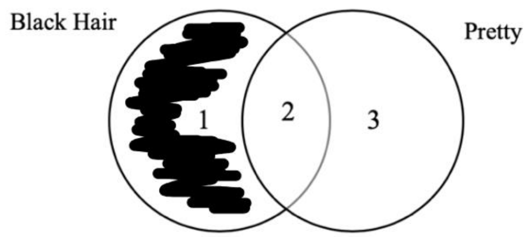


Aristotle was a philosopher who lived in Greece from 384-322 B.C. Socrates taught Plato, Plato taught Aristotle, and Aristotle taught Alexander the Great. Aristotle is known for his great contributions to the field of zoology, including animal dissections, but is primarily known for his contributions to philosophy. In his teaching, he used both deductive and inductive reasoning.

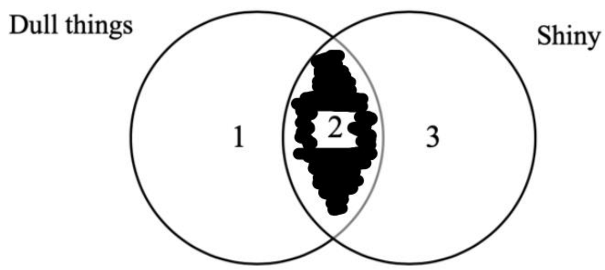
Aristotle believed the world had no beginning or end. Genesis 1:1 says, “In the beginning, God created the heavens and the earth.” Therefore, many believe our world did have a beginning as it was created by God. Moreover, John 1:1, “In the beginning was the Word, and the Word was with God, and the Word was God.” Deductive reasoning here leads one from the general to the specific. This could also be a syllogism whose premise leads to a VALID conclusion.

Aristotle also enjoyed working on syllogisms with students at his school in Athens, Greece. In these syllogisms, when the word "all" was used in the premise, it must be used in the conclusion. If the premise used "no" or "nothing," then the conclusion must also use the word "no" or "nothing." If the premise used the word "some," then the conclusion must also use the word "some."

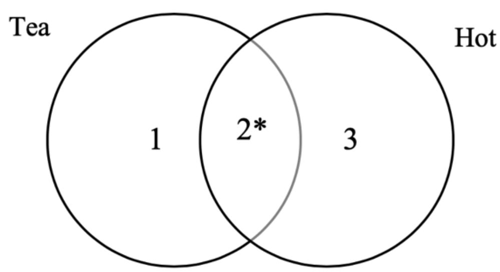
When the word "all" is used, the area *outside* the overlap is shaded. Example: If all black hair is pretty, then there is no black hair that is not pretty.



When the word "no" is used, the area *inside* the overlap is shaded. Example: If no shiny things are dull, then all dull things are not shiny.



When the word "some" is used, *nothing* is shaded. Example: If some tea is hot, some tea is not hot. Put an asterisk (*) by "2" to show that "some" tea is hot (and so, consequently, "some" is not hot), but do not shade anything.



Section 3.12 Conditional Statements and Truth Tables

Looking Back 3.12

A conditional statement is like a syllogism, but rather than a premise and argument, a conditional statement includes a hypothesis and conclusion. It is of the form, "If p, then q (written "p ⟶ q")." Unlike a syllogism, it is in lowercase letters rather than capital letters. However, it is like a syllogism in that the hypothesis may be a false statement.

Conditional statements are used in geometry to draw conclusions. The "if" part of the statement is the hypothesis, and the "then" part of the statement is the conclusion.

Example 1: Rewrite each statement as a conditional statement using the form "if-then."

a) All cats have fur.

b) You are in math class when you are learning geometry.

"If p, then q" can be written using the symbol "⟶" (p ⟶ q), which may be read, "p implies q."

"All 112° angles are obtuse angles" is a true statement. It can be written as a conditional statement: "If an angle is 112°, then it is an obtuse angle." However, if the statement read, "All obtuse angles are 112°," it could be false. An obtuse angle could be 112°, but it could also be 103° or 96°. All three of these angles are obtuse. The conditional statement, "If an angle is obtuse, then it is 112°" leads to a false conclusion when the hypothesis is true, but the conclusion is false.

If both p and q are false in the statement, the conditional statement could still be true. An example of this is the statement, "If an angle is not 112°, then the angle is not an obtuse angle." The angle could be 38° and that is an acute angle. A conditional is true as long as there are no cases in which the hypothesis is true, but the consequence is false. If the hypothesis is not true, then no judgment can be made about the conclusion.

For example, suppose an advertisement says that if you pay to rush an order you will receive it in two days. Look at the four possible outcomes and decide in which case the advertisement was false:

1) You pay to rush an order and the order arrives in two days.
2) You pay to rush an order and it does not arrive in two days.
3) You do not pay to rush an order and it arrives in two days.
4) You do not pay to rush an order and it does not arrive in two days.

Example 2: Complete the truth table for the conditional statements.

Conditional		
p	**q**	**p ⟶ q**
T	T	
T	F	
F	T	
F	F	

Note: A conditional is considered true in traditional logic as long as there are no cases in which the hypothesis is true, and the conclusion is false. Syllogisms are not always considered traditional logic.

Looking Ahead 3.12

For any conditional there are also three related statements: the converse, the inverse, and the contrapositive. We will investigate these in the following examples.

Let p be "two lines do not intersect," and let q be "two lines are parallel" for Example 3, 4, and 5. The conditional statement $p \longrightarrow q$ represents the statement: "If two lines do not intersect, the two lines are parallel." This could be a false statement as the two lines could be skew.

Example 3: The converse of a conditional statement exchanges the hypothesis and conclusion. "If p, then q ($p \longrightarrow q$)" becomes "if q, then p ($q \longrightarrow p$)."

Write the converse of the conditional statement "If two lines do not intersect, the two lines are parallel" from above and tell whether it is true or false.

Example 4: The negation of a statement is the opposite of the original statement. Negate a) and b) below. The negation of a conditional statement is the opposite of the hypothesis and the opposite of the conclusion. The symbol for negation ($\sim$) comes in front of the letters representing the hypothesis and conclusion. "If p, then q ($p \longrightarrow q$)" becomes "If not p, then not q ($\sim p \longrightarrow \sim q$)." This is the inverse of the original statement. To find the inverse, negate both the hypothesis and conclusion.

Find the inverse of the conditional statements for c) and d) and e) below and tell whether they are true or false.

a) Kenny is tall.

b) Jonah has black hair.

c) If you are in Cincinnati, you are in Ohio.

d) If you are in Ohio, you are in Cincinnati.

e) If two lines do not intersect, the two lines are parallel.

Note: It is logical to reason that both the converse and inverse of a conditional statement are either both true or, both false. The converse and the inverse are said to be logically equivalent.

Example 5: The contrapositive of a conditional statement is the negation of the hypothesis and conclusion of the converse of a conditional statement. This comes in the form "If not q, then not p ($\sim q \longrightarrow \sim p$)."

Write the contrapositive of the conditional statement: "If two lines do not intersect, the two lines are parallel." State if the contrapositive is true or false.

Note: It is logical to reason that a conditional statement and its contrapositive are both true or both false. The conditional and the contrapositive are said to be logically equivalent. This means they share the same truth table.

Example 6: Given the truth table for the conditional, complete the table for the converse, inverse, and contrapositive.

		Conditional	**Converse**	**Inverse**	**Contrapositive**
p	q	$p \to q$	$q \to p$	$\sim q \to \sim p$	$\sim q \to \sim p$
T	T	T			
T	F	F			
F	T	T			
F	F	T			

Example 7: A biconditional statement is a single statement that uses the phrase, "if and only if," such as, "two lines are perpendicular if and only if they meet to form a right angle." It is written using the symbol "↔" ($p \leftrightarrow q$) which may be read "p if and only if q."

Let the hypothesis be "two angles are complementary." Let the conclusion be "two angles have measures with a sum of 90°."

Write the biconditional statement.

Note: A biconditional is a logical conditional statement in which the hypothesis and conclusion are interchangeable.

Section 3.13 Formal Geometric Proofs

Looking Back 3.13

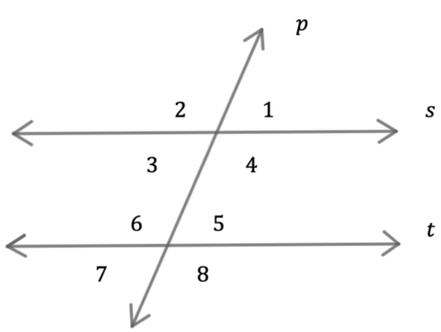
A formal proof uses deductive reasoning and logical arguments to demonstrate that a statement is true. As you have previously seen, a two-column formal proof shows statements on the left side and corresponding reasons (which are logical arguments) on the right side. These are numbered in logical order. The left-hand column usually begins with given information and proceeds to other statements by applying known or accepted axioms, postulates, properties, or theorems.

You first learned about the reflexive, symmetric, and transitive properties in Pre-Algebra. Now, they can be applied to segments and angles in Geometry as well. Then you can use them to prove other theorems.

Example 1: a) Given $\overline{AB}$, apply the Reflexive Property of Segment Congruence.
b) Given $\overline{AC} \cong \overline{BD}$, apply the Symmetric Property of Segment Congruence.
c) Given that $\overline{AC} \cong \overline{BD}$ and $\overline{BD} \cong \overline{FE}$, apply the Transitive Property of Segment Congruence.

When you used the tracing paper in the Practice Problems for Section 3.5, you unknowingly used the Translation Postulate, which you will learn about when we study the symmetry of polygons. The Translation Postulate states that a slide is rigid motion (which means the transformation keeps the size and shape of the figure). Now, we will use what you learned about corresponding angles and vertical angles to prove that Alternate Interior Angles are equal. We will not demonstrate the proof for all theorems but will use proven theorems in other proofs.

Example 2: Prove that Alternate Interior Angles are congruent.
Given: $s \parallel t$
Prove: $\angle 3 \cong \angle 5$



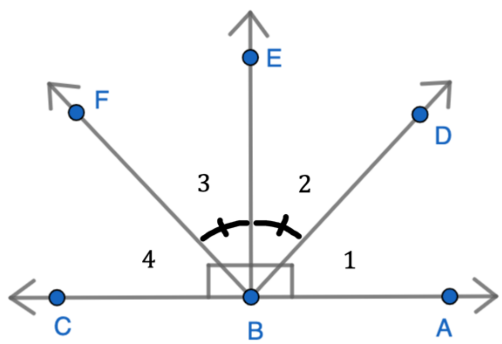
You used the Rotation Postulate, which states that the figure is in rigid motion, to prove that Alternate Interior Angles are congruent. Complete the reasons for the statements below.

Statements	**Reasons**
1. $s \parallel t$	1. Given
2. $\angle 3 \cong \angle 7$	
3. $\angle 5 \cong \angle 7$	
4. $\angle 3 \cong \angle 5$	

Example 3:	Write the proof for the other pair of alternate interior angles in Example 2. Given: $s \parallel t$ Prove: $\angle 4 \cong \angle 6$

Statements	Reasons
1. $s \parallel t$	1. Given
2.	
3.	
4.	

Example 4:	Given: $m\angle 2 = m\angle 3$ Prove: $m\angle 1 = m\angle 4$



Statement	Reason
1. $m\angle 2 = m\angle 3$	1.
2. $m\angle ABE = 90°$	2.
3. $m\angle CBE = 90°$	3.
4. $m\angle ABE = m\angle 1 + m\angle 2$	4.
5. $m\angle CBE = m\angle 3 + m\angle 4$	5.
6. $m\angle 1 + m\angle 2 = m\angle 3 + m\angle 4$	6.
7. $m\angle 1 + m\angle 2 = m\angle 2 + m\angle 4$	7.
8. $m\angle 1 = m\angle 4$	8.