1- Title of my conference

Lie-Rinehart algebras and applications in differential geometry

2- Abstract of my conference

Hereafter A denotes a commutative algebra with unit element 1_A, over a commutative field K with characteristic *zero*. The case when A is not a field is essential.

We denote by $\mathrm{Der}_K(A)$ the A-module of all derivations of A and by $\mathrm{Diff}_K(A)$ the A-module of all differential operators of order ≤ 1 of A, M denotes a C^∞ differentiable paracompact and connected manifold, $C^\infty(M)$ the real algebra of numerical C^∞ functions on M, $\mathfrak{X}(M)$ the $C^\infty(M)$-module of vector fields on M and $\mathcal{D}(M)$ the $C^\infty(M)$-module of differential operators of order ≤ 1 on $C^\infty(M)$.

We consider a K-Lie algebra $\mathcal{G}$, with bracket $[,]$, endowed with an A-module structure. At the beginning, a Lie-Rinehart algebra has been defined as a pair $(\mathcal{G}, \rho)$ where

$$\rho : \mathcal{G} \longrightarrow \mathrm{Der}_K(A)$$

is simultaneously a morphism of A-modules and of K-Lie algebras satisfying

$$[x, a \cdot y] = \rho(x)(a) \cdot y + a \cdot [x, y]$$

for any $x, y \in \mathcal{G}$ and any $a \in A$.

Now, in differential geometry, this definition can be considered as the algebraic version of a Lie algebroid.

The main goal of my conference is the study of Lie-Rinehart algebras in my sense of and their applications in differential geometry.

Henceforth a Lie-Rinehart algebra over A or an A-Lie-Rinehart algebra is a pair $(\mathcal{G}, \rho)$ where

$$\rho : \mathcal{G} \longrightarrow \mathrm{Diff}_K(A)$$

is simultaneously a morphism of A-modules and of K-Lie algebras, satisfying

$$[x, a \cdot y] = [\rho(x)(a) - a \cdot \rho(x)(1_A)] \cdot y + a \cdot [x, y]$$

for any $x, y \in \mathcal{G}$ and any $a \in A$.

By denoting by $\mathcal{L}^*_{sks}(\mathcal{G}, A)$ the graded module of skew-symmetric A-multilinear forms on $\mathcal{G}$ and by d_ρ the coboundary operator associated with the representation ρ, the pair $(\mathcal{L}^*_{sks}(\mathcal{G}, A), d_\rho)$ is a differential algebra: it is called the differential algebra of the Lie-Rinehart algebra $(\mathcal{G}, \rho)$.

We say that A is a Jacobi algebra when A is endowed with a Lie algebra structure with bracket $\{,\}$ such that, for any $a \in A$, the inner derivation

$$ad(a) : A \longrightarrow A, b \longmapsto \{a, b\} ,$$

is a differential operator of order ≤ 1 of the commutative algebra A. When for any $a \in A$, $ad(a)$ is a derivation of the commutative algebra A, in this case we say that A is a Poisson algebra.

A manifold M is said to be a Jacobi manifold (respectively is said to be a Poisson manifold) when the real algebra $C^\infty(M)$ is a Jacobi algebra (respectively is a Poisson algebra).

When A is a Jacobi algebra, the A-module $A \times \Omega_K(A)$, where $\Omega_K(A)$ is the A-module of Kähler differential, carries a Lie-Rinehart algebra structure. The cohomology algebra of this Lie-Rinehart algebra is the cohomology of the Jacobi algebra A. We define in the same manner the cohomology of a Poisson algebra.

When A is any commutative algebra over K, a characterization of Jacobi algebras in terms of Lie-Rinehart algebras structures on the A-module $A \times \Omega_K(A)$ is given.

Another manner to give a characterization of Jacobi algebras or Poisson algebras is the vanishing of the Schouten-Nijenhuis bracket of Jacobi 2-form or of Poisson 2-form. This other characterization is also given.

A triple $(\mathcal{G}, \rho, \omega)$ is a symplectic Lie-Rinehart algebra when the pair $(\mathcal{G}, \rho)$ is a Lie-Rinehart algebra and

$$\omega : \mathcal{G} \times \mathcal{G} \longrightarrow A$$

is a skew-symmetric nondegenerate bilinear form such that

$$d_\rho \omega = 0.$$

To say that the skew-symmetric bilinear form ω is nondegenerate means that the map

$$\mathcal{G} \longrightarrow \mathcal{G}^*, x \longmapsto i(x)(\omega),$$

is an isomorphism of A-modules, where $\mathcal{G}^*$ is the A-module of linear forms on $\mathcal{G}$ and

$$i(x)(\omega) : \mathcal{G} \longrightarrow A, y \longmapsto \omega(x, y).$$

We give the relationship between Schouten-Nijenhuis algebras and graded Lie algebras. For example when V is a K-vector space and when $End_K(V)$ is the K-Lie algebra of all endomorphisms of V, for any $f\ , g \in End_K(V)$, the brachet

$$[f, g] = f \circ g - g \circ f$$

is the usual Lie algebra bracket on $End_K(V)$. We will see that the Schouten-Nijenhuis bracket of f and g is

$$[f, g]_{SN} = g \circ f - f \circ g$$

the opposite of the usual Lie algebra bracket on $End_K(V)$.

We also show that a manifold M is a locally conformal symplectic manifold if and only if the $C^\infty(M)$-module of vector fields on M, $\mathfrak{X}(M)$, admits a symplectic Lie-Rinehart algebra structure.

We also show that a manifold M is a contact manifold if and only if the $C^\infty(M)$-module of differential operators of order ≤ 1 on $C^\infty(M)$, $\mathcal{D}(M)$, carries a symplectic Lie-Rinehart algebra structure.

We study the symplectic Lie-Rinehart algebra structure on the module of all global sections of a Lie algebroid. That generalizes the case of the tangent bundle TM and the case of the vector bundle $\mathbb{R} \times TM$.

We give a generalization of the Lie-Rinehart algebra notion. The Frölicher-Nijenhuis bracket is also generalized, so and the Nijenhuis tensor.

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